Exact Fractal Area of Two-Dimensional Vesicles

In a Letter[1] Leibler, Singh, and Fisher studied the thermodynamic behavior of two-dimensional (2D) vesicles (see also Ref. 2). The latter are described by a bead-and-tether model of a closed self-avoiding membrane, which also embodies a pressure difference $\Delta p = p_{\text{int}} - p_{\text{ext}}$ between the interior and the exterior. The number of beads is N. In a stress ensemble, Δp is fixed and the area of a vesicle, A, fluctuates subject to a Boltzmann weight $\exp(\Delta p A/k_B T)$. In the *flaccid* case where $\Delta p = 0$, the model reduces to closed self-avoiding walks (SAW). By Monte Carlo methods they measured the mean square radius of gyration $\langle R_G^2 \rangle \approx R_0^2 N^{2\nu}$ as $N \to \infty$, finding $\nu = 0.755 \pm 0.018$, in agreement with the exact $\nu = \frac{3}{4}$.[3] Now,[2] does this fractal fill out its interior so that $\langle A \rangle / R_G^2 \to 0$, or does it have an area proportional to the square of its linear dimension so that if $\langle A \rangle \approx A_0 \times N^{2\nu_A}$, one has $\nu_A = \nu$? The above authors found[1] $\nu_A/\nu = 1.007 \pm 0.013$, thus conjecturing the equality.[4] In this Comment we show that the conjecture $\nu_A = \nu = \frac{3}{4}$ is true and contained in results recently obtained for the 2D O(n) model.[5]

The latter is defined, e.g., on a hexagonal lattice, by the partition function $Z_{O(n)} = \sum_{\mathcal{G}} K^N n^L$, where the sum extends over all configurations $\mathcal{G}$ of nonintersecting loops, with a total number of bonds N and of loops L. For $n \to 0$, $\partial Z_{O(n)}/\partial n|_{n=0}$ describes the partition function of a single loop, with fluctuating length. The critical point is[3] $K_c = [2 + (2-n)^{1/2}]^{-1/2}$ for $n \in [-2, 2]$, where the loops become infinite and fractal. For $K > K_c$ one has a new *dense* critical phase, where the infinite loops *fill* the plane. Let us call D the fractal dimension of the connected domains inside the loops and D_H that of the loops (hulls). Then the linear size of a large N-bond loop scales as $\langle R^2 \rangle \sim N^{2\nu_H}$, where $\nu_H \equiv 1/D_H$ (to be distinguished for $n \neq 0$ from the correlation-length exponent ν), and the area $\langle A \rangle$ of the domains scales as $\langle A \rangle \sim \langle R \rangle^D \sim N^{D\nu_H}$. Hence $\langle A \rangle \sim N^{2\nu_A}$ gives $\nu_A \equiv \frac{1}{2} D\nu_H$. For polymers $\nu_H = \nu$, and the above conjecture $\nu_A = \nu$ actually corresponds to the interior fractal dimension $D = 2$. In Ref. 5, D was calculated by a Coulomb-gas method as a function of $n \in [0,2]$, $n = -2\cos(\pi g)$, as $D = 1 + \frac{1}{2} g + 3/8g$, where $g \in [1, \frac{3}{2}]$ or $g \in [\frac{1}{2}, 1]$ for the dilute or dense phase, respectively. One also has[6] $D_H = 1/\nu_H = 1 + 1/2g$. For a self-avoiding closed walk ($n = 0$, $g = \frac{3}{2}$, dilute, or $g = \frac{1}{2}$, dense) we find $D = 2$ in both phases. $D(n)$ (Fig. 1) varies *very little* for $n \in [0,2]$ ranging from $D = 2$ to 1.875 for $n = 2$ (XY model) and passing in the lower *dense* branch through the percolation-cluster fractal dimension $D = \frac{91}{48} = 1.8958\ldots$ for $n = 1$, and in the upper *dilute* branch through $D = \frac{187}{96} = 1.9479\ldots$ for the $n = 1$ Ising clusters.[5,7]

Dense polymer loops also have $D = 2$. Since $\nu_H = 1/d = \frac{1}{2}$, we get $\langle A \rangle \sim N^{D\nu_H} \sim N$. The area then scales as the length. The universality class and topology are then

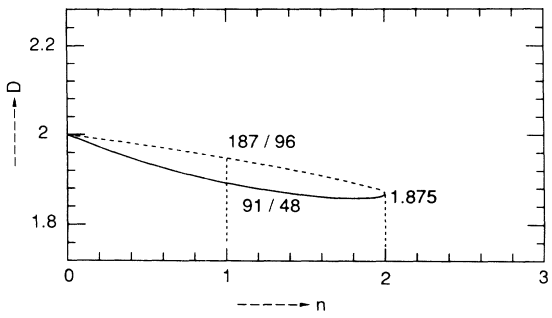

FIG. 1. Fractal dimension of the connected domains in the critical O(n) model (dotted line) or in the dense phase (solid line). Only for $n = 0$, do we have $D = 2$.

those of *spanning trees*[6] (ST) with, essentially, each arm formed of a double chain, the tree being close packed. In the *deflated* regime ($\Delta p < 0$) Leibler, Singh, and Fisher[1] also observe $\langle A \rangle \sim N$ but in the different universality class of branched polymers (BP) with $\nu \approx 0.65 \pm 0.04 > \frac{1}{2}$. Actually spanning trees are *collapsed* branched polymers, obtained by attraction between the arms, an effect which *cannot* be created by a pressure deflation. Another route to ST, i.e., dense polymers, would be in a strain ensemble to enclose the vesicle in a box of crushing area. Whence the series of fixed points of increasing stability for 2D vesicles, with their central charges:[8] flaccid (SAW, $c = 0$) → deflated (BP) → crushed (ST, $c = -2$). By the c theorem[9] (if valid for these nonunitary systems), c decreases along the renormalization flow. Hence the (unknown) central charge c of BP or animals should satisfy $-2 \leq c \leq 0$.

Bertrand Duplantier
Service de Physique Théorique de Saclay
91191 Gif-Sur-Yvette CEDEX, France

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