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Lattice Gas as a Model of $1/f$ Noise

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Direct numerical measurement of the power spectrum of the number of particles on the lattice demonstrates in an example of probably broad physical relevance that $1/f$ behavior can arise due to self-organized criticality. Different versions of the model are studied in order to look for universality.

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The widespread occurrence of signals exhibiting power spectra with behavior over many frequency decades described by a power law with an exponent close to -1 suggests that an underlying explanation might exist. The concept of self-organized criticality, recently introduced by Bak, Tang, and Wiesenfeld,[1] is an attempt to find such a general principle. The idea is that the $1/f$-like power spectra occur when a many-body system organizes itself in a dynamical state which is critical in the sense that no characteristic length or time scale exist. The ubiquity of the $1/f$ phenomena follows once it has been established that the self-organized critical state is a general property of most dissipative many-body systems.

In principle, molecular-dynamics simulations should be able to shed some light on these ideas, but since the issue is the low frequency, i.e., the long-time behavior, it does not seem possible to overcome the numerical demands for the large system size of interest. The next best thing is to study discrete dynamics. Bak, Tang, and Wiesenfeld[1] used a cellular automaton with a threshold. New experiments by Jaeger, Liu, and Nagel[2] have made the physical applicability of this cellular-automaton model somewhat unclear. Furthermore, recent numerical analysis of the power spectra of the model[3] shows that the model behaves as $1/f^2$ rather than $1/f^\beta$ with $\beta\sim 1$.

In this Letter we present a lattice-gas model with a simple qualitative physical connection to transport phenomena such as flux flow in type-II superconductors[4] or highway traffic[5] which are known to contain $1/f$ power spectra. The physical quantity which in our lattice gas show $1/f$ behavior is the total number of particles on the lattice, $N(t)$. Particles can enter the system randomly at the left edge of the lattice. While on the lattice the particles interact with neighbors according to a deterministic equation of motion. The interaction between the particles makes them perform a random walk. Particles leave the system when they return to the left edge or reach the right edge of the lattice.

The power spectra of $N(t)$ are found to behave as $S(f)\sim 1/f^\beta$ with $\beta\sim 1.5$ in one dimension and $\beta\sim 1.2$ in two dimensions. The scaling behavior of the power spectra are connected with the power-law behavior of the distribution of lifetimes,[1,6] $D(T)\sim 1/T^\alpha$ (T being the time the particles spend on the lattice), where α and β approximately fulfill $\alpha+\beta=3$.

The model.—Our lattice-gas model was inspired by the experiment on flux flow in thin-film type-II superconductors performed by Yeh and Kao.[4] We want to model particles which follow diffusive dynamics. The equation of motion we have in mind is of the form $\eta\mathbf{v}=\mathbf{F}$, where η is a friction coefficient, $\mathbf{v}$ is the velocity, and $\mathbf{F}$ is the total force on the particle. The model is defined as follows: Consider a lattice of $N_x\times N_y$ sites. Each site can contain one or zero particles. Particles on neighbor sites repel each other with a central force of unit strength. Let $\mathbf{f}_{\rm par}$ denote the total force on a particle due to its neighbor particles. An additional driving force, $\mathbf{f}_{\rm dr}$, can be applied to all the particles. Let $\mathbf{F}$ be the total force on a particle, i.e., $\mathbf{F}=\mathbf{f}_{\rm par}+\mathbf{f}_{\rm dr}$, and define the vector $\mathbf{n}$ as

$$n_x=[F_x/F] \text{ and } n_y=[F_y/F], \tag{1}$$

where $[t]$ is the integer nearest to t. A particle on site $\mathbf{r}_0$

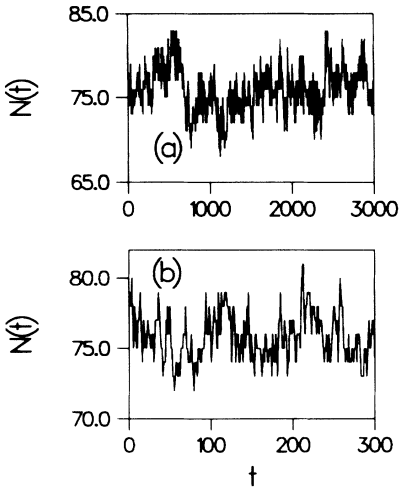

FIG. 1. Time sequence of the number of particles on the lattice for a system of size 50×6, with NN interaction, the density of pinning sites equal to $\frac{1}{3}$, and no driving force applied. (b) A blow up of a section of (a).

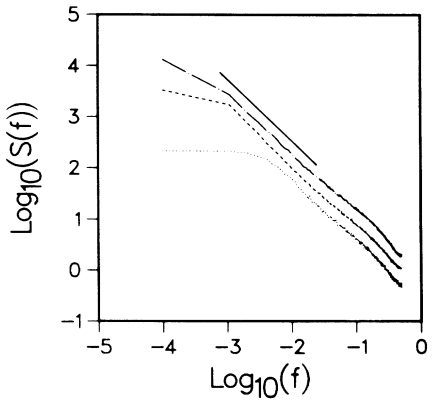

FIG. 2. Power spectra of systems with NNN interaction, the density of pinning sites equal to $\frac{1}{3}$, and no driving force applied. Three different system sizes are shown: dotted line corresponds to 20×6; dashed line to 50×6; and dash-dotted line to 100×6. The data have been multiplied by factors of 1, 4, and 8—going from bottom to top—in order to keep the curves apart. The straight line is $1/f^{1.2}$.

is moved to site $\mathbf{r}_1$ given by

$$\mathbf{r}_1 = \mathbf{r}_0 + \mathbf{n} \tag{2}$$

if the new site $\mathbf{r}_1$ is unoccupied; otherwise, the particle is left on site $\mathbf{r}_0$. The whole lattice is simultaneously updated. In case two or more particles want to move into the same new site, the one with the largest force wins. The sites $(0,y)$ at the left edge of the lattice are unoccupied with fixed particles. This rim of particles tends to push particles on the sites $(1,y)$ into the lattice. In each time step particles in the column $(1,y)$ are first removed and then new particles are introduced on the sites $(1,y)$ with a probability p per site.[7] Particles can freely leave the system over the right rim. The two-dimensional systems are made periodic in the y direction. We have considered systems of size $N_x = 100$, 500, 1000, 2000, and 5000 in one dimension. In two dimensions we used $N_x = 10$, 20, 50, 100, and 250 and $N_y = 6$ or 18, for the study of temporal features of the system, as well as systems of sizes 30×30, 60×60, and 128×128 for the spatial characterization discussed below.

Furthermore, we can choose a subset of sites which we denote as pinning centers. If a particle sits on such a site, it will only be allowed to move off the site when $A_p F > 1$, where A_p is a number characterizing the strength of the pinning.

The particle density and the mean velocity depend on the values of p, A_p, and the density of pinning sites n_p. However, the qualitative features and the critical exponents do not depend on the specific values of these parameters. All simulations presented here are run with $p = 0.2$, $A_p = 1$ or 0.9, and $n_p = 0$ or $\frac{1}{3}$.

The simulations are started out with some configuration of particles on the lattice (empty lattice, every second site occupied, random distribution of particles, etc.). After a certain transient time the system enters a statistically stationary state which is independent of the initial configuration. The total number of particles $N(t)$ is then recorded as a function of time. Figure 1 shows a typical measurement of $N(t)$ in the stationary state. In Fig. 1(b) we show a magnification of a section of Fig. 1(a) in order to exhibit the self-similar structure of the $N(t)$ curve.

Results.—The power spectrum is obtained by direct Fourier transformation of $N(t)$. In order to achieve sufficient statistics many power spectra of successive time sequences are averaged. During the simulation we measure the time the particles stay on the lattice from which we construct the lifetime distribution $D(T)$. Figures 2 and 3 show typical measured spectra and lifetime

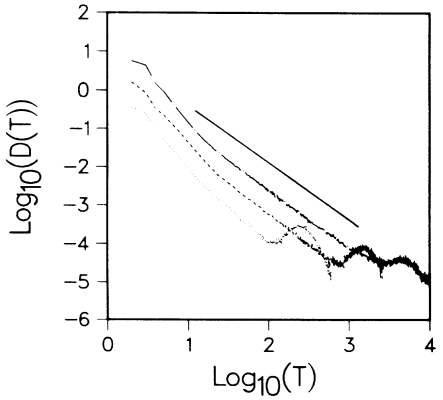

FIG. 3. Lifetime distributions corresponding to the power spectra shown in Fig. 2. The same signatures and multiplications are used as in Fig. 2. The straight line is $1/T^{1.5}$.

distributions. Results for different system sizes are shown. The size dependence is discussed below.

We have studied four different versions of the model in order to address the question to what extent universality exists as in ordinary thermodynamical critical phenomena: nearest-neighbor (NN) interaction with no driving force and no pinning centers in one dimension, and nearest- and next-nearest-neighbor (NNN) interactions with or without driving force and pinning centers. The results are summarized in Table I. The α and β exponents depend on the dimension of the system. Within the numerical accuracy they do not depend on the number of neighbor interactions nor does the introduction of pinning sites influence the value of the exponents. However, a small applied driving force makes the exponents change.

Some remarks about the case of a finite driving force are appropriate. For forces of strength $f_{dr} < f_{cr}$ the driving force has no effect on the depinning of particles ($f_{cr} = -1 + 1/A_p$ for $A_p \simeq 1$). The only effect of the driving force is to make particles with no neighbors move to the right; these particles do not move when $f_{dr} = 0$. (This is the reason for the change in the σ exponent, see below.) The α and β exponents for $0 < f_{dr} < f_{cr}$ are given in Table I. As f_{dr} becomes larger than f_{cr} the force starts to aid the depinning of pinned particles and to change the direction of motion of particles interacting with neighboring particles. When f_{dr} is the dominating force the particles are pulled over the lattice by moving one step to the right in each time step; i.e., the speed is equal to 1. The distribution of lifetimes becomes a delta function, $D(T) = \delta(T - N_x)$, and the power spectrum becomes proportional to[3] $\sin^2(\pi N_x f)/f^2$. This behavior is consistent with the experiment by Yeh and Kao,[4] who found a crossover to a Lorentzian-like form well above threshold.

A random linear superposition of signals with a lifetime distribution $D(T) \propto 1/T^{\alpha}$ leads to a power spectrum[3] $S(f) \propto 1/f^{3-\alpha}$ (when $\alpha \in [1,3]$). Table I shows that this relation is satisfied approximately for the α and β exponents of the present model. Although the recorded power spectrum is measured directly on a highly interacting model, the critical state can be characterized by a set of independent lifetimes. Temporal correlations appear to be of only minor importance. In this way the model offers important support to the original ideas of Bak, Tang, and Wiesenfeld.[1]

TABLE I. Exponents for different versions of the model. The α, β, and γ exponents are determined with an accuracy of about 5%. The accuracy of σ is about 2%. The exponents are defined in the text.

Dimension	Range of interaction	Density of pinning sites	f_{dr}	α	β	γ	σ
1	NN	0	0	1.5	1.5	0.6	2
2	NN	0	0	1.7	1.1	2.2	2
2	NNN	0	0	1.5	1.2	2.2	2
2	NNN	$\frac{1}{3}$	0	1.5	1.2	2.1	2
2	NNN	$\frac{1}{3}$	0.01	1.8	1.3	1.9	1

Table I contains an exponent σ which we are now going to discuss. The main part of the straight-line segment in Fig. 3 comes from particles which manage to diffuse back to the left input edge of the system. The net flow in the system is to the right. The center of mass, or average velocity, along the x axis, $\langle v_x \rangle$, is positive.[8] The time $T_0 = N_x/\langle v_x \rangle$ it takes the center of mass of the particle system to travel the distance N_x across the lattice is found to scale as $T_0 \propto N_x^{\sigma}$. When no driving force is applied σ is equal to 2, i.e., ideal diffusion, whereas a nonzero f_{dr} changes σ to 1. The local peak at long times (see Fig. 3) is located about T_0. This suggests that $D(T)$ should fulfill the following finite-size scaling form: $D(T) = T^{-\alpha} F(T/N_x^{\sigma})$. However, this finite-size scaling ansatz turns out to be fulfilled only poorly. A much more satisfying fit is obtained by use of the multifractal form suggested by Kadanoff *et al.*,[9]

$$\frac{\ln[D(T,N_x)]}{\ln(N_x/N^*)} = f\left(\frac{\ln(T/T^*)}{\ln(N_x/N^*)}\right). \tag{3}$$

Figure 4 shows a multifractal scaling plot of the lifetime distribution in Fig. 3. The values of the fitting parameters T^* and N^* are here both equal to 1. Multifractal scaling has also been observed in other models[9,10] or self-organized criticality. We shall get back to the value of σ below, but before that it will be convenient to consider the spatial characteristics of the system.

The system is obviously anisotropic due to the boundary conditions. There is a density gradient along the x axis. In one dimension the density drops rapidly close to the $x = 1$ edge. Throughout the rest of the system the density is almost constant. The density profile is different in two dimensions. For all the cases with $f_{dr} = 0$ the density decreases linearly with constant slope from $x \simeq 1$ to $x \simeq N_x$. Application of the driving force in the

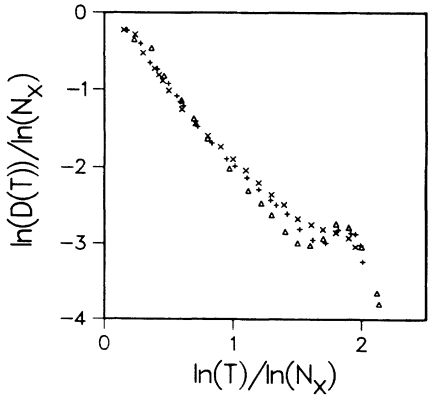

FIG. 4. Multifractal scaling plot of $D(T)$ in Fig. 3. System sizes are as follows: △, 20×6; +, 50×6; and ×, 100×6.

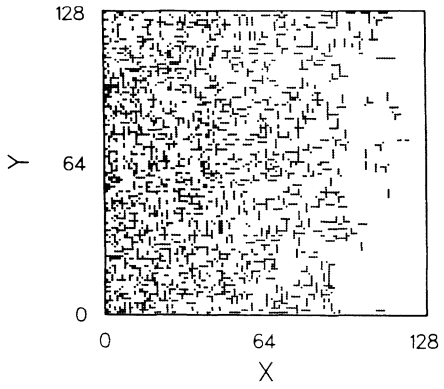

FIG. 5. Clusters of energy-dissipating sites in a system with NN interaction, without pinning sites, and no applied driving force.

presence of pinning sites changes the density profile. For $f_{dr}>0$ the density drop takes place close to the edges $x=1$ and $x=N_x$, whereas the density gradient in the bulk is small.

The density profile and the σ exponent can be discussed in terms of a diffusion equation. Since the particles are conserved except at the boundary, the particle density (averaged along the y direction) $n(x)$ fulfills $dn/dt=-dJ/dx$, where J is the particle current. The updating algorithm suggests that the current has the form

$$J=-a(dn/dx)+bf_{dr}n\,, \tag{4}$$

where a and b are coefficients. The static density profile is readily obtained and expressed in terms of the densities close to the edges $n(0)=n_0$ and $n(N_x)=n_L$:

$$n(x)=\frac{n_0\exp(bf_{dr}N_x/a)-n_L-(n_0-n_L)\exp(bf_{dr}x/a)}{\exp(bf_{dr}N_x/a)-1}\,. \tag{5}$$

For $f_{dr}=0$ this represents a linearly decreasing density with a slope which becomes numerically smaller for finite f_{dr}. From Eqs. (4) and (5) we get

$$J=a\frac{n_0-n_L}{N_x}+b\frac{n_0+n_L}{2}f_{dr} \tag{6}$$

for small f_{dr}. The edge densities n_0 and n_L are determined by the condition at the boundary and are independent of the system size. Since $\langle v_x\rangle=J/\bar{n}$, where $\bar{n}$ is the average density, we find that $\langle v_x\rangle\sim 1/N_x$ for $f_{dr}=0$ and $\langle v_x\rangle$ is independent of the system size when $f_{dr}>0$. This explains the behavior of the exponent σ.

Let us look at energy dissipation in the model. The updating algorithm models an overdamped equation of motion. We can think of energy being dissipated every time a particle is moved from one site to another. In order to study the spatial distribution of this energy dissipation we mark all the sites from which particles leave at time step t and mark in addition all the sites these particles move onto at time $t+1$. These are the sites on which energy has been dissipated during the update from t to $t+1$. The sites marked in this way form clusters containing different numbers S of sites. The distribution of cluster sizes $D(S)$ follows a power law, $D(S)\sim 1/S^{\gamma}$. The exponent γ is listed in Table I. Figure 5 shows an example of the clusters for a system with NN interaction, no pinning sites, and $E_{dr}=0$.

We have presented a physically reasonable discrete-time model consisting of particles moving around on a lattice. At zero or small driving force the model contains a self-organized critical state in which distribution functions and power spectrum have power-law behavior. The model offers solid support to the idea that $1/f$ behavior can arise as a result of self-organized criticality.

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[6]For a discussion of the connection between the distribution of lifetimes of the individual signals and the power spectrum of a time signal consisting of a random superposition of these elementary signals, see Jensen, Christensen, and Fogedby (Ref. 3).

[7]In the one-dimensional case the updating of the site $x=1$ is done in a slightly different way. Particles on site 1 are only removed with a probability p. If the site $x=1$ is unoccupied, a new particle is introduced with probability p.

[8]We calculate $\langle v_x\rangle$ as the increment in the x direction during one time step averaged over the particles present at that time step and then average over all time steps made.

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[10]C. Tang (to be published).