Dipole and Monopole Vortices in Nonlinear Drift Waves

K. H. Spatschek, E. W. Laedke, Chr. Marquardt, S. Musher,[a] and H. Wenk
Institut für Theoretische Physik I, Heinrich-Heine-Universität Düsseldorf, D-4000 Düsseldorf, Federal Republic of Germany
(Received 30 January 1990)

The problems of existence and stability of drift vortices in plasmas with density and temperature inhomogeneities are studied analytically and numerically. It is shown that basically two space scales are involved: (i) the ion Larmor radius at the electron temperature, where dipolar vortices exist, and (ii) a longer space scale on which monopolar vortices are possible. The latter are shown to survive finite disturbances. A Liapunov functional for structural stability is presented. The analytical predictions are verified by a 2D numerical simulation.

PACS numbers: 52.35.Kt, 52.35.Mw, 52.35.Ra

The fundamental work of Hasegawa and Mima[1] on the existence and nonlinear dynamics of dipolar drift vortices in plasmas with density inhomogeneities stimulated many investigations on spatially coherent nonlinear structures in drift-wave turbulence.[2] Various authors[2-4] emphasized that solitary vortices could play an important role besides the mode-mode-interaction processes.[5-7] Meanwhile, there exist experimental verifications[8,9] of some of the predictions which also reveal a great similarity to astrophysical observations.[10] Also, in numerics,[2] dipolar as well as monopolar structures are seen. With respect to the latter, Petviashivili[11] developed a model which later on was criticized by some authors.[12,13] Thus the situation is as follows: For dipolar drift vortices in plasmas with density inhomogeneity, the Hasegawa-Mima equation[1] is the well-accepted correct model; the potential varies on the characteristic length scale ρ_s, the ion Larmor radius at the electron temperature. However, the problem of temperature-gradient model equations is still open. In this paper we want to contribute to a resolution of the controversial discussion in the literature[11-14] by applying a multiple-scale analysis. The first main outcome of this discussion will be that the existence of dipolar or monopolar vortices, respectively, depends on the length scales under consideration: Dipolar vortices exist on the ρ_s scale, whereas the characteristic length of the possible monopolar vortices is of the order ρ_s/ϵ, where ϵ is a smallness parameter to be discussed later. The second result of our investigation was stimulated by Su *et al.*[3] These authors predicted numerically a structural destabilization of dipolar vortices. Thus we asked ourselves the question of whether the predicted monopoles are structurally stable. The answer is yes and the proof will be given in the paper. Finally, we have developed a numerical solver for (spatially) two-dimensional Hasegawa-Mima-type equations in order to test the stability predictions. The advantages and disadvantages of various procedures are discussed and numerical results are presented.

Let us exemplify these ideas for a simple model; more complicated generalizations are possible along the lines outlined below. We assume a uniform external magnetic field $B\hat{\mathbf{z}}$, write the ion density (n_i) continuity equation in the simplest form,

$$(\partial_t + \mathbf{v}_{i\perp}\cdot\nabla)\ln n_i + \nabla\cdot\mathbf{v}_i = 0\,, \tag{1}$$

and assume quasineutrality and Boltzmann-distributed electrons,

$$n_i \approx n_e \approx n_0 \exp(e\varphi/k_B T_e)\,. \tag{2}$$

Here, φ is the electrostatic potential and T_e is the electron temperature. For the 2D ion velocity we use the drift approximation

$$\mathbf{v}_{i\perp} \approx \mathbf{v}_{\mathbf{E}\times\mathbf{B}} - (c/\Omega_i B)(\partial_t + \mathbf{v}_{\mathbf{E}\times\mathbf{B}}\cdot\nabla)\nabla\varphi\,, \tag{3}$$

where $\Omega_i = eB/cm_i$ and $\mathbf{v}_{\mathbf{E}\times\mathbf{B}} = (c/B)\hat{\mathbf{z}}\times\nabla\varphi$.

A straightforward combination of these equations leads to a single, however, complicated, equation for φ where, of course, the background density n_0 and the electron temperature are space dependent, e.g., $n_0 = n_0(x)$ and $T_e = T_e(x)$. In order to simplify Eqs. (1)-(3) to a tractable model equation it is most appropriate to introduce within a multiple-scale analysis the variables $x_i = \epsilon^i x$, $\eta_i = \epsilon^i \eta$, $t_i = \epsilon^i t$, for $i \geq 0$ and $\epsilon \ll 1$. Here, we moved into the frame $\eta = y - ut$.

The well-known Hasegawa-Mima equation[1] is obtained in the scaling $\phi = \epsilon\phi_1(x_0,x_1,x_2,\ldots,\eta_0,\eta_1,\eta_2,\ldots,t_1,t_2,\ldots) + \epsilon^2\phi_2 + \cdots$, $n_0 = n_0(x_1,x_2,\ldots)$, $u = u_1 \sim 0(\epsilon)$, and $T_e = T_e(x_1,x_2,\ldots)$. Measuring T_e in T_0 $(T = T_e/T_0)$, φ in $k_B T_0/e$ $(\phi = e\varphi/k_B T_0)$, $v_{i\perp}$ in c_s $(v = v_{i\perp}/c_s)$, t in Ω_i^{-1} $(t\Omega_i \to t)$, and $r_\perp$ in ρ_s $(x/\rho_s \to x,\ \eta/\rho_s \to \eta)$, we can write the result as

$$\frac{\partial}{\partial t_1}\left[\frac{1}{T(x_1)} - \nabla_0^2\right]\phi_1 - \{\phi_1, \nabla_0^2\phi_1\}_0 + u_1\frac{\partial}{\partial\eta_0}\nabla_0^2\phi_1 - u_1\left(\frac{1}{T(x_1)} + \frac{\kappa_n(x_1)}{u_1}\right)\frac{\partial}{\partial\eta_0}\phi_1 = 0\,. \tag{4}$$

Here, ∇_0^2 is the 2D Nabla operator for the coordinates x_0 and η_0, $\{,\}_0$ is the Poisson bracket, also with respect to x_0 and η_0, and $\kappa_n = \partial_x \ln n_0$. Note that the Hasegawa-Mima equation determines the variation of ϕ_1 on the x_0,η_0,t_1 scales. Then all the x_1 dependences, e.g., of κ_n and T, are irrelevant, meaning that the coefficients T^{-1} and κ_n/u_1 are constant on the x_0 scale.

It is also easy to see that other scalings exist which allow monopolar solutions. The simplest one is $\phi=\epsilon^2\phi_2(x_1,x_2,\ldots,\eta_1,\eta_2,\ldots,t_5,t_6,\ldots,)+\epsilon^3\phi_3+\cdots$, $\eta_0=n_0(x_2,x_3,\ldots)$, $u=u_2\sim O(\epsilon^2)$, $T=T(x_2,x_3,\ldots)$, and $T^{-1}+\kappa_n/u_2\sim O(\epsilon^2)$. In this case one obtains in the lowest relevant order (appropriate for large-scale structures) the equation

$$\frac{\partial}{\partial t_5}\frac{\phi_2}{T(x_2)}+u_2\frac{\partial}{\partial\eta_1}\nabla_\mathrm{I}^2\phi_2-u_2\left[\frac{1}{T(x_2)}+\frac{\kappa_n(x_2)}{u_2}\right]_2\frac{\partial}{\partial\eta_1}\phi_2+\kappa_T(x_2)\phi_2\frac{\partial}{\partial\eta_1}\phi_2=0\,. \tag{5}$$

Here, ∇_I^2 is the 2D Nabla operator for the coordinates x_1 and η_1, $\kappa_T=T^{-1}\partial\ln T/\partial x$, and $[\cdots]_2$ indicates that the terms in the brackets combine to a second-order (in ϵ) contribution. Several aspects are worth mentioning: (i) In this model, the potential depends on x_1 and η_1. Thus the characteristic length scale is ρ_s/ϵ and not ρ_s as is true for dipolar vortices. (ii) Although the temperature is space dependent, the coefficient $\kappa_T(x_2)$ can be considered as constant when ϕ_2 is solved as a function of x_1 and η_1. Thus the multiple-scale analysis resolves some of the controversies in the literature.[13] (iii) The nonlinear term originates due to temperature inhomogeneity. However, because of the required relation

$$\frac{1}{T(x_2)}+\frac{\kappa_n(x_2)}{u_2}=\epsilon^2 f(x_2,x_3,\ldots)\,, \tag{6}$$

obviously, after differentiation with respect to x_2,

$$\kappa_T=\kappa_n'/u_2 \tag{7}$$

follows which has been required by Lakhin, Mikhailovskii, and Onochenko[12] by different arguments. In our opinion, the above arguments make the physical origin of the scalar nonlinearity clearer.

Obviously, there exist some intermediate scalings where both the vector nonlinearity $\{\phi,\nabla^2\phi\}$ and the scalar nonlinearity $\kappa_T\phi\partial_\eta\phi$ appear on the same footing. [A trivial example is $\phi=\epsilon^2\phi_2(x_1,x_2,\ldots)$, $n_0=n_0(x_3,x_4,\ldots)$, $T=T(x_3,x_4,\ldots,)$, $u=u_3\sim O(\epsilon^3)$, $T^{-1}+\kappa_n/u\sim O(\epsilon^2)$.] This is important in one respect. The model equations (4) and (5) are valid in restricted parameter regimes. Thus, not only does the stability problem exist in the sense of initial perturbations,[14] but also *structural* perturbations can appear. For example, the dynamics of the dipolar solutions of Eq. (4) should be investigated when a structural perturbation in the form of a scalar nonlinearity is present[3] and the dynamics of the monopolar vortices of Eq. (5) should be discussed when a structural perturbation in the form of a vector nonlinearity is present. The first problem was already considered numerically[3] with the interesting result that dipolar vortices are destroyed. More work is in progress[15] on this topic. Here, we solve the second problem: We show that monopolar vortices are quite stable. This fact will be proved analytically, and also demonstrated numerically, in the following.

Let us start with the analytical part. We present the example of the structural stability of a monopole solution when it is disturbed by a vector-type nonlinearity. In this case, the basic equation is

$$\partial_t\phi+u\partial_\eta\nabla^2\phi-u\rho^2\partial_\eta\phi+\kappa_T\phi\partial_\eta\phi=\alpha\{\phi,\nabla^2\phi\}\,, \tag{8}$$

where the right-hand side is considered as the (structural) disturbance. For simplicity we have omitted all indices, introduced a smallness parameter α, and used the abbreviation $\rho^2:=1+\kappa_n/u$. Equation (8) has stationary monopolar solutions which are shown in Fig. 1. The parameter $\Omega=\kappa_T/2u\rho^2$ controls the form of their radial

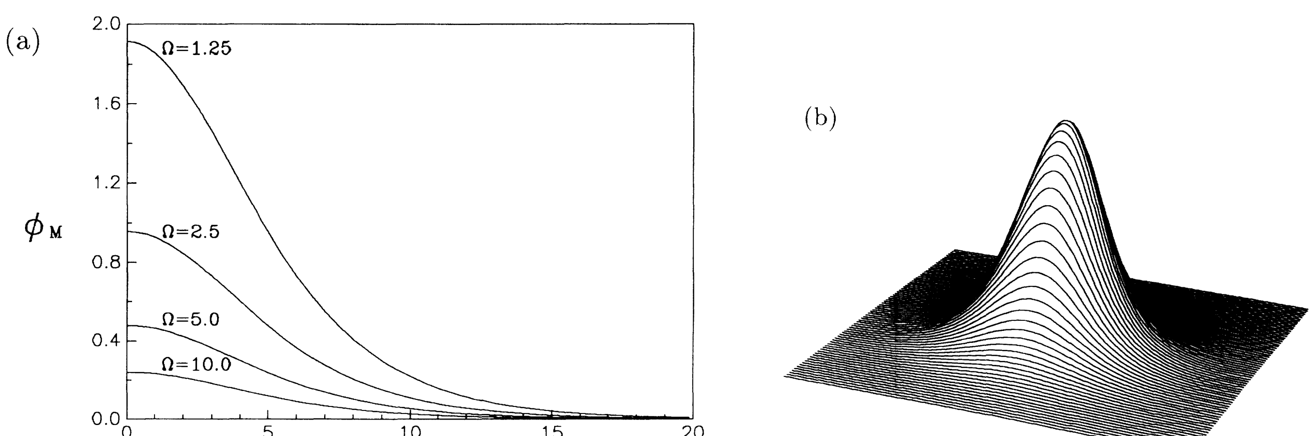

FIG. 1. Numerical solutions for stationary monopoles. They are obtained from Eq. (8) for $\partial_t=0$ and $\phi=\phi(r)$. The common parameters are $\rho^2=0.09$ and $u=0.11$. (a) Radial dependence of the monopole ϕ_M for different parameters Ω. (b) 3D plot of ϕ_M for $\Omega=2.5$.

dependences. After multiplying with ϕ and $\kappa_T\phi^2$, respectively, and subsequent integration over space, we obtain conserved quantities which can be combined to

$$L := \int d^2r \left[(\nabla\phi)^2 - \frac{1}{3}\frac{\kappa_T}{u}\phi^3 + \rho^2\phi^2 \right] . \tag{9}$$

Now, $\tilde{L} = L\{\phi\} - L\{\bar{\phi}_M\}$ can be considered as a Liapunov functional.[16] Here, $\bar{\phi}_M$ is the reference state from the invariant set S which is generated from the stationary monopole ϕ_M under consideration by translations, $\bar{\phi}_M = \phi_M(\mathbf{r}-\boldsymbol{\xi})$. In $\tilde{L}$, $\bar{\phi}_M$ is defined as the element of S which is closest to ϕ. We have to note that with $\phi_M(r)$, $\bar{\phi}_M(x-\xi_x, \eta-\xi_\eta)$ is a stationary solution of Eq. (8).

For conservative perturbations, with $\int d^2r\,\phi^2 = \int d^2r\,\phi_M^2$, we have $\delta\tilde{L} = 0$, and for the second variation of $\tilde{L}$ we obtain

$$\delta^2 L = \int d^2r\, \delta\phi H \delta\phi , \tag{10}$$

where $H = -\nabla^2 - (\kappa_T/u)\bar{\phi}_M + \rho^2$. Because of the discreteness of the eigenvalues $\lambda < \rho^2$ of the operator H there exists a positive constant $p > 0$ with $\langle\psi|H|\psi\rangle > p\langle\psi|\psi\rangle$, provided the function ψ fulfills the relations $\langle\psi|\bar{\phi}_M\rangle = 0$, $\langle\psi|\partial_x\bar{\phi}_M\rangle = 0$, and $\langle\psi|\partial_\eta\bar{\phi}_M\rangle = 0$. The proofs of these facts follow by variational procedures. The constraints $\langle\psi|\partial_x\bar{\phi}_M\rangle = 0$ and $\langle\psi|\partial_\eta\bar{\phi}_M\rangle = 0$ follow from the consistency relations when the closest state $\bar{\phi}_M$ is determined. For nonconservative perturbations, with $\langle\psi|\bar{\phi}_M\rangle \neq 0$, we can prove stability with respect to an intermediate state $\tilde{\phi}_M$ which is close to $\bar{\phi}_M$. Thus the stability of monopoles with respect to initial as well as structural perturbations is proved.

For the numerical investigations of vortices we developed two different numerical schemes: a semi-implicit Crank-Nicholson-type algorithm with operator splitting[15] and an explicit leapfrog scheme. Both codes are supplemented by a fast elliptic solver for the solution of the vorticity equation at each time step, and both are of second-order accuracy in time and space. The details of the numerical methods will be presented elsewhere.[15]

The codes were tested by monitoring several conserved quantities during the time development of, e.g., a dipole as an initial distribution in the Hasegawa-Mima case. These quantities remained constant with a relative accuracy $\lesssim 10^{-4}$ in long-time runs ($t \approx 400\Omega_i^{-1}$).

The numerics verified very precisely the structural stability of monopolar solutions. This is important since the analytic predictions, although valid for finite perturbations, are restricted to small disturbances. In Fig. 2(a) we show the final state of an initial monopole at $t = 400\Omega_i^{-1}$ for $u = -0.11$, $\rho^2 = 0.09$, and $\kappa_T = -0.05$ ($\Omega = 2.5$). By comparing with Fig. 1(b) we cannot recognize any destabilizing tendency. Even the more detailed diagnostic, as depicted in Fig. 2(b), completely supports the structural stability of monopolar vortices. The same behavior occurs when we perturb the initial distribution.

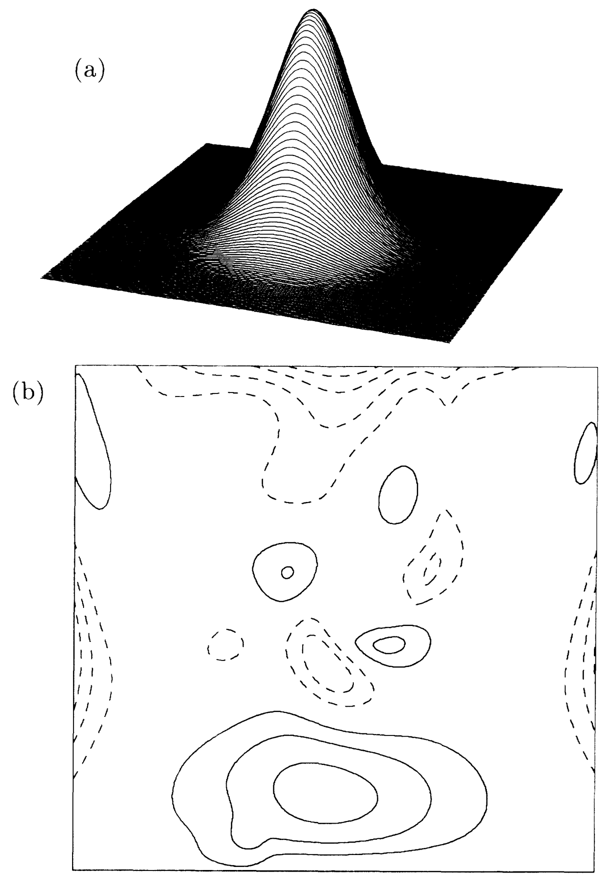

FIG. 2. (a) Time development for an initial monopole with $u = -0.11$, $\kappa_T = -0.05$, and $\rho^2 = 0.09$. The 3D plot is for $t = 400\Omega_i^{-1}$. (b) To support the stability result we have plotted the pointwise differences $\phi_M(t=400) - \phi_M(t=0)$ for $-20 \leq x, \eta \leq 20$. The contour lines show positive (——) and negative (---) deviations in steps $\Delta\phi_M = 0.75 \times 10^{-3}$.

These results also throw some new light on the structural destabilization of dipolar vortices.[3] When starting with a dipolar-vortex solution of the Hasegawa-Mima equation and structurally perturbing the latter, we observe the destabilization. For example, we add to the right-hand side of Eq. (4) the term $-\kappa_T\phi_1(\partial\phi_1/\partial\eta_0)$ and solve for the parameter values $T = 1$, $\kappa_n = 0.1$, $u_1 = -0.15$, $a = 6$, and $\kappa_T = -0.05$. On the other hand, we can also use Eq. (8) with $\alpha = 1$. Figure 3 shows, in addition, the tendency to form stable monopolarlike structures which will survive for a long time. This simulation clearly supports our conjecture that monopolar structures are extremely important in drift-wave turbulence.

In conclusion, for vortices scaling on the ion Larmor radius at the electron temperature and weak temperature as well as density inhomogeneities, the Hasegawa-Mima equation[1] is the correct model. A different situation occurs when we look for vortices on a long scale compared with the ion Larmor radius at the electron temperature. Then monopolar structures are possible. The

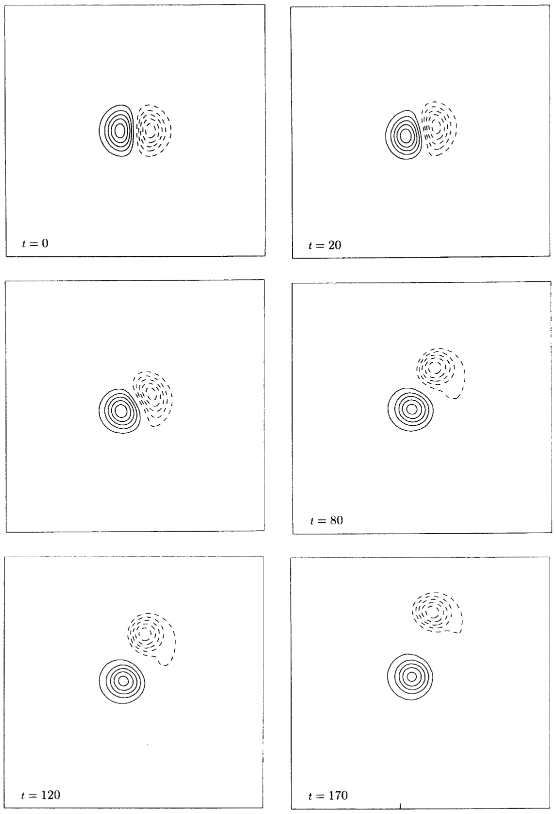

FIG. 3. Breakup of a dipolar vortex (shown by its contour lines in steps of $\Delta\phi_D = 0.5$ in x-η space) when a structural perturbation is added to the Hasegawa-Mima equation. The tendency to form monopolar structures is clearly seen $(-25 \leq x, \eta \leq 25)$.

coefficient of the scalar nonlinearity is proportional to the temperature gradient. This is not in contradiction to the work of Lakhin, Mikhailovskii, and Onochenko[12] since consistency requires $\kappa_T \approx \kappa_n'/u$. Most important is the new result that the monopolar vortices are quite stable coherent structures. In contrast to the dipolar vortices, they are structurally stable. This conclusion was obtained by analytical tools and is supported by 2D numerics. The agreement between the analytical predictions and the numerical computations is excellent. In the future, the applicability of the 2D approximation will be also discussed in the light of some new developments.[17]

This work was supported by the Deutsche Forschungsgemeinschaft. The numerical assistance by V. Naulin is gratefully acknowledged.

[a]Permanent address: Institute of Automation and Electrometry, Siberian Division of the U.S.S.R. Academy of Sciences, 630090 Novosibirsk, U.S.S.R.

[1]A. Hasegawa and K. Mima, Phys. Rev. Lett. **39**, 105 (1977); Phys. Fluids **21**, 87 (1978).

[2]W. Horton, Phys. Fluids B **1**, 537 (1989).

[3]X. Su, W. Horton, P. J. Morrison, and V. I. Pavlenko, University of Texas, Austin, report, 1989 (to be published).

[4]M. Kono and E. Miyashita, Phys. Fluids **31**, 326 (1988).

[5]P. W. Terry and W. Horton, Phys. Fluids **25**, 491 (1982).

[6]D. Biskamp and He Kaifen, Phys. Fluids **28**, 2172 (1985).

[7]A. Hasegawa and Y. Kodama, Phys. Rev. Lett. **41**, 1470 (1978); A. Hasegawa, C. G. Maclennan, and Y. Kodama, Phys. Fluids **22**, 2122 (1979).

[8]S. V. Antipov, M. V. Nezlin, V. K. Rodionov, A. Yu. Rylov, E. N. Snezhkin, A. S. Trubnikov, and A. V. Khutoretskii, Sov. J. Plasma Phys. **14**, 648 (1988).

[9]K. Saeki and Y. Amagishi, in Proceedings of the International Conference on Plasma Physics, Delhi, 1989, edited by A. Sen and P. K. Kaw, p. 953.

[10]M. V. Nezlin, Usp. Fiz. Nauk **150**, 3 (1986) [Sov. Phys. Usp. **29**, 807 (1986)].

[11]V. I. Petviashivili, Fiz. Plazmy **3**, 270 (1977) [Sov. J. Plasma Phys. **3**, 150 (1977)]; Pis'ma Zh. Eksp. Teor. Fiz. **32**, 628 (1980) [JETP Lett. **32**, 614 (1980)].

[12]V. P. Lakhin, A. B. Mikhailovskii, and O. G. Onochenko, Phys. Lett. A **119**, 348 (1987); Plasma Phys. Controlled Fusion **30**, 457 (1988).

[13]J. Nycander, Phys. Fluids B **1**, 1796 (1989).

[14]E. W. Laedke and K. H. Spatschek, Phys. Fluids **29**, 133 (1986); **31**, 1492 (1988).

[15]Chr. Marquardt, S. Musher, K. H. Spatschek, and H. Wenk (to be published).

[16]D. D. Holm, J. E. Marsden, T. Ratiu, and A. Weinstein, Phys. Rep. **123**, 1 (1985).

[17]M. N. Rosenbluth and R. N. Sudan, Phys. Fluids **29**, 2347 (1986).