Calculation of Universal Scaling Function for Free-Electron-Laser Gain

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We present a variational calculation of free-electron-laser gain in the exponential regime before saturation using a dispersion relation incorporating the energy spread, emittance, and focusing of the electron beam, and the diffraction and guiding of the radiation. Rapid computation facilitates free-electron-laser design optimization. Results are expressed in a universal scaled form.

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The free electron laser (FEL) holds great promise[1] as a source of intense coherent radiation at wavelengths below 1000 Å. In the case of an FEL oscillator, the difficulty in developing high-reflectivity mirrors for short-wavelength radiation requires high gain to overcome large cavity losses. For a single-pass FEL, the *e*-folding length of the amplified radiation must be minimized to reduce the required length of the wiggler magnet. To achieve high gain for short-wavelength radiation, the electron beam must have high peak current, small normalized transverse emittance, and small energy spread. Strong focusing of the electron beam is also necessary to achieve the shortest wavelengths, and the use of ion focusing has been proposed.[2]

In this Letter, we present an analytic calculation of the FEL gain valid in the regime of exponential growth before saturation, based upon a dispersion relation[3,4] derived from the Vlasov-Maxwell equations. This extends earlier analytical work on the exponential regime by including, simultaneously, the effects of the energy spread, emittance, and the focusing[5] of the electron beam, and the diffraction and guiding[6-8] of the radiation field. The dispersion relation is solved using a variational approximation,[9] and the results for the *e*-folding length of the electric field of the fundamental guided mode are expressed in a scaled form [Eq. (11)]. With a reasonable choice of the frequency detuning [Eq. (13)], the *e*-folding length is a universal function of three dimensionless variables (emittance-to-wavelength ratio, scaled electron focusing strength, and scaled electron-energy spread) and a dimensionless scaling parameter [Eq. (10)] measuring transverse current. Some graphs of the universal function are given in Figs. 1(a)-1(c). These graphs can be used for quick estimates of the gain for quite general parameters. Until now, only large simulation codes[10,11] requiring long CPU running times on fast computers could include all the effects determining FEL gain. Our analytic approach has several advantages: rapid computation time, elucidation of the dependence of the gain on the large number of system parameters, and the ease of design optimization. We have checked that the variational approximation yields agreement to within 1% with exact results[12] which we have derived for a parallel electron beam with finite beam size and energy spread, and to within 5% with several cases with nonvanishing emittance run on the simulation codes FRED and FELEX. These comparisons will be described elsewhere.

In our calculation, the electron beam's energy distribution $h(\gamma)$ is Gaussian, with average energy $\gamma_0 mc^2$, and rms spread $\gamma_0\sigma$. The static wiggler magnetic field has period length λ_w and wave number $k_w = 2\pi/\lambda_w$. The resonant radiation frequency $\omega_r = k_r c$ of the FEL is determined by $k_r = 2\gamma_0^2 k_w/(1+K^2)$, where $K = eB_{\text{rms}}/k_w mc$ and B_{rms} is the rms value of the wiggler magnetic field (in mks units). We assume either a helical wiggler or a flat wiggler with parabolic pole faces. After averaging over the fast wiggler oscillations, the transverse electron motion is described by harmonic betatron oscillations[5] in the transverse displacement, $d^2\mathbf{R}/dz^2 = -k_\beta^2 \mathbf{R}$. In the absence of external focusing, the betatron wave number[5] is $k_\beta = Kk_w/\gamma\sqrt{2}$.

Initially, we assume the electron beam has a uniform longitudinal density, and a uniform "water-bag" distribution inside a four-dimensional sphere in the four-dimensional transverse phase space $\mathbf{R} = (x,y)$, $\mathbf{R}' = d\mathbf{R}/dz = (x',y')$:

$$U(\mathbf{R},\mathbf{R}') = \frac{n_0}{\pi k_\beta^2 R_0^2}\Theta(k_\beta^2 R_0^2 - k_\beta^2 R^2 - R'^2)\,, \qquad (1)$$

where the step function $\Theta(v) = 1$ for $v > 0$ and $\Theta(v) = 0$ for $v < 0$. Integrating $U(\mathbf{R},\mathbf{R}')$ over the angular deviation $\mathbf{R}'$, one obtains the parabolic transverse density profile: $g(\mathbf{R}) = n_0(1 - R^2/R_0^2)$ for $R < R_0$, and $g(\mathbf{R}) = 0$ for $R > R_0$. The peak electron density is n_0, and the electron-beam current is $I_0 = en_0 c\pi R_0^2/2$. The rms transverse emittance ϵ of the matched electron beam is defined by

$$\epsilon = (\langle x^2\rangle\langle x'^2\rangle)^{1/2} = (\langle y^2\rangle\langle y'^2\rangle)^{1/2} = k_\beta R_0^2/6\,. \qquad (2)$$

We consider the linear region before saturation, and

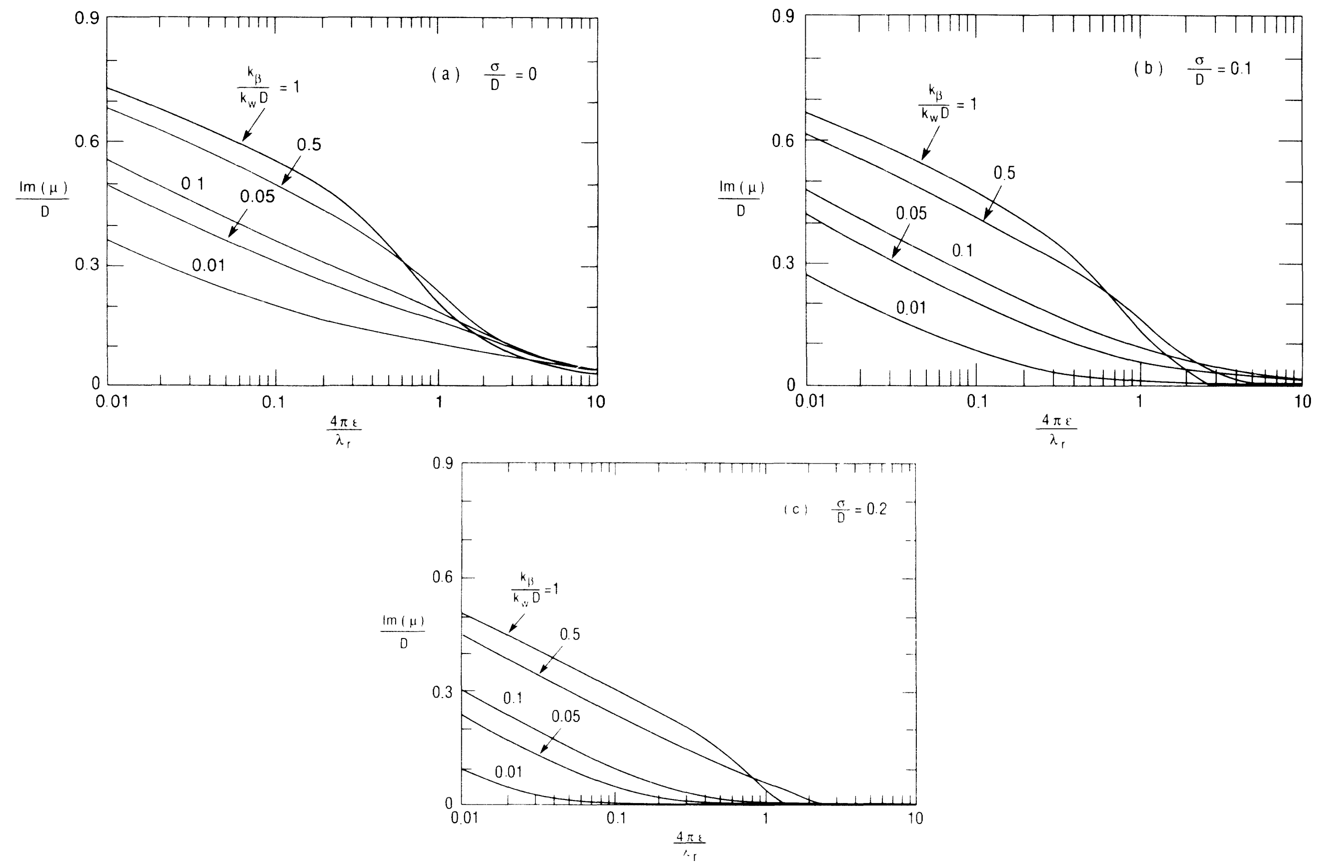

FIG. 1. Scaling function vs scaled emittance for several values of $k_\beta/k_w D$, corresponding to scaled energy spreads (a) $\sigma/D=0$, (b) $\sigma/D=0.1$, and (c) $\sigma/D=0.2$, with the detuning as given in Eq. (13).

write the electric field of the fundamental guided laser mode of frequency ω in the form

$$E(\mathbf{r},\omega)e^{-i\mu k_w z}e^{-i\omega(t-z/c)}\hat{\mathbf{e}}+\text{c.c.}, \tag{3}$$

where $\hat{\mathbf{e}}$ is either the helical-polarization vector $(\hat{\mathbf{x}}+i\hat{\mathbf{y}})/\sqrt{2}$ or the linear-polarization vector $\hat{\mathbf{x}}$. The function $E(\mathbf{r},\omega)$ describes the transverse-mode profile in terms of the dimensionless coordinates[13] $\mathbf{r}=\sqrt{2k_r k_w}\mathbf{R}$. The factor $\exp(-i\mu k_w z)$ describes the deviation from free-space propagation and $2\pi\,\mathrm{Im}\mu$ is the growth rate per wiggler period. The e-folding length L of the electric field of the amplified mode is given by

$$1/L=k_w\,\mathrm{Im}\mu\,. \tag{4}$$

Assuming $\gamma_0\gg 1$, so that space-charge effects are negligible, Vlasov-Maxwell equations have been used to derive a dispersion relation[3] determining μ and $E(\mathbf{r})$:

$$(\nabla_\perp^2+\mu)E(\mathbf{r})=\frac{i}{2}(2\rho\gamma_0)^3\int\frac{d\gamma}{\gamma^2}h'(\gamma)\int d^2p\int_{-\infty}^{0}ds\,e^{-ias}u(p^2+\kappa^2r^2)E\left(\mathbf{r}\cos(\kappa s)+\frac{\mathbf{p}}{\kappa}\sin(\kappa s)\right), \tag{5}$$

where $\kappa=k_\beta/k_w$ characterizes the focusing strength and the Laplacian $\nabla_\perp^2$ corresponds to the dimensionless coordinates $\mathbf{r}$. In addition, we have defined

$$a=\mu+(\omega-\omega_r)/\omega_r-2(\gamma-\gamma_0)/\gamma_0+(p^2+\kappa^2r^2)/4\,,$$

$$u(p^2+\kappa^2r^2)=(1/\pi\kappa^2a^2)\theta(\kappa^2a^2-\kappa^2r^2-p^2)\,,$$

$$(2\rho\gamma_0)^3=e^2Z_0n_0K^2[JJ]^2/2mck_w^2\,,$$

with $[JJ]=1$ for a helical wiggler, and

$$[JJ]=J_0(K^2/2(1+K^2))-J_1(K^2/2(1+K^2))$$

for a flat wiggler. The radius corresponding to the edge of the electron beam expressed in dimensionless coordinates is

$a=\sqrt{2k_r k_w}R_0$, and $Z_0=377\ \Omega$ is the impedance of free space.

The dispersion relation of Eq. (5) corresponds to the stationary solutions of the variational form

$$\int d^2r\,E(\mathbf{r})(\nabla_\perp^2+\mu)E(\mathbf{r})=\int d^2r\int d^2r'\,E(\mathbf{r})K(\mathbf{r},\mathbf{r}')E(\mathbf{r}')\,, \tag{6}$$

with

$$K(\mathbf{r},\mathbf{r}')=(2\rho)^3\int h(\gamma)d\gamma\int_{-\infty}^{0}\frac{\kappa^2 s\,ds}{\sin^2(\kappa s)}u(\kappa^2a^2-w)e^{isw/4}\exp\left\{-is\left[\mu+\frac{\omega-\omega_r}{\omega_r}-2\left(\frac{\gamma-\gamma_0}{\gamma_0}\right)+\frac{\kappa^2a^2}{4}\right]\right\},$$

where

$$w=\frac{\kappa^2}{\sin^2(\kappa s)}[a^2\sin^2(\kappa s)-r^2-r'^2+2\mathbf{r}\cdot\mathbf{r}'\cos(\kappa s)]\,.$$

The region of integration over $d^2r\,d^2r'$ in Eq. (6) is restricted to $w\geq 0$. As a trail function, we choose

$$E(\mathbf{r})=\begin{cases}e^{-\chi r^2/2a^2}, & r\leq a\,,\\ AH_0^{(1)}(r\sqrt{\mu}), & r\geq a\,,\end{cases} \tag{7}$$

where we require $\mathrm{Im}\sqrt{\mu}>0$ to satisfy the boundary condition at $r\to\infty$, and $H_0^{(1)}$ is the Hankel function. Continuity of the logarithmic derivative at $r=a$ leads to the constraint

$$a\sqrt{\mu}H_0^{(1)\prime}(a\sqrt{\mu})/H_0^{(1)}(a\sqrt{\mu})=-\chi\,. \tag{8}$$

The trial function of Eq. (7) is the exact solution[6] outside the electron beam, and inside it has the correct leading behavior in both the large and small electron-beam-size limits. Although the trial function contains no free parameters to vary, the stationary property of the variational form assures that the error in the complex eigenvalue μ depends quadratically on small errors in the trial function. Employing Eq. (7) in the variational form of Eq. (6) yields

$$Da^2\left(\frac{\mu}{d}\right)(1-e^{-\chi})-\chi[1-(1-\chi)e^{-\chi}]$$
$$=\int_{-\infty}^{0}\frac{s\,ds}{\cos(\kappa s/D)}\exp\left[-i\left(\frac{\mu}{D}+\frac{\omega-\omega_r}{\omega_r D}\right)s-2\left(\frac{\sigma}{D}\right)^2 s^2\right]\left(\frac{1-e^{-\eta_+}}{\eta_+}-\frac{1-e^{-\eta_-}}{\eta_-}\right), \tag{9}$$

with

$$\eta_\pm=3is\left(\frac{\kappa}{D}\right)(k_r\epsilon)+\frac{\chi}{2}\left[1\mp\cos\left(\frac{\kappa}{D}s\right)\right],$$

$$D=\left(\frac{4eZ_0}{\pi mc^2}\frac{K^2}{1+K^2}\frac{I_0}{\gamma_0}\right)^{1/2}[JJ]\,. \tag{10}$$

Equations (8) and (9) can be solved to determine complex parameters χ and μ/D. Observing that $Da^2=12k_r\times\epsilon(D/\kappa)$, it is seen that the e-folding length L of the electric field can be expressed in the scaled form

$$\frac{1}{k_wL}=\mathrm{Im}\mu=DG\left(k_r\epsilon,\frac{\sigma}{D},\frac{k_\beta}{k_wD},\frac{\omega-\omega_r}{\omega_rD}\right). \tag{11}$$

The scaling parameter D defined in Eq. (10) is a measure of the transverse electron current. It is straightforward to verify that the scaling noted in Eq. (11) also holds for the exact solution of the dispersion relation of Eq. (5). Therefore, this scaling should provide a useful way to organize the results of computer-simulation codes.

An alternate form of the scaling relation is

$$\frac{1}{k_wL}=\mathrm{Im}\mu=\rho F\left(\tilde{a},\frac{\sigma}{\rho},\frac{k_\beta}{k_w\rho},\frac{\omega-\omega_r}{\omega_r\rho}\right), \tag{12}$$

which follows immediately from Eq. (11) upon noting that $D=2\rho\tilde{a}$, where ρ is the Pierce parameter[13] defined following Eq. (5) and the scaled electron-beam radius $\tilde{a}$ is defined by[14] $\tilde{a}^2=2\rho a^2=24(k_r\epsilon)(\rho k_w/k_\beta)$.

The one-dimensional calculation of Ref. 13 assumes a parallel electron beam and ignores diffraction of the radiation. Angular spread was included by Colson and Blau,[15] but diffraction was ignored. Moore[6] included diffraction, but not angular spread. Our paper presents the first analytic calculation including all of the effects of energy spread, emittance, and focusing of the electron beam, and the diffraction and guiding of the radiation. The importance of dimensionless scaling variables was discussed in Refs. 1 and 2. Our scaling relations of Eqs. (11) and (12) are new results, and here we present the first calculation of the universal scaling function determining FEL gain.

Utilizing the variational approximation, we have determined the universal gain function $G=\mathrm{Im}\mu/D$ [of Eq. (11)], by numerically solving Eqs. (8) and (9). Because of the scaling law, the entire physical parameter space can be described by the dependence of G on four dimensionless scaled variables. In Figs. 1(a)-1(c) we plot $G=\mathrm{Im}\mu/D$ against $2k_r\epsilon=4\pi\epsilon/\lambda_r$, for several values of the scaled focusing strength k_β/k_wD, corresponding to scaled energy spread (a) $\sigma/D=0$, (b) $\sigma/D=0.1$, and (c) $\sigma/D=0.2$. For each point $(2k_r\epsilon,k_\beta/k_wD,\sigma/D)$, the

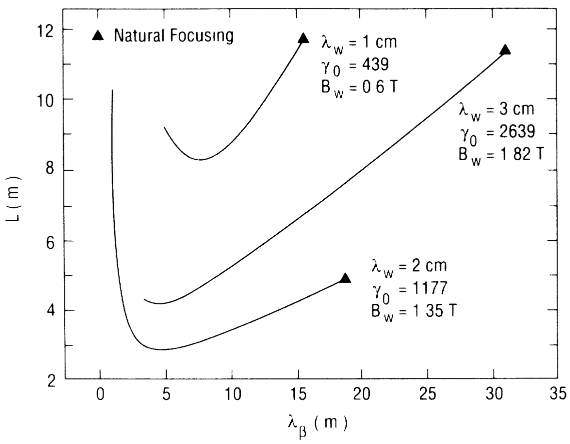

FIG. 2. Electric-field e-folding length at 300 Å vs betatron wavelength for wiggler periods of 1, 2, and 3 cm.

scaled detuning is chosen to be

$$\frac{\omega-\omega_r}{\omega_r D}=-3\left(\frac{k_\beta}{k_w D}\right)k_r\epsilon\,, \qquad (13)$$

which we have found to yield near-maximum gain. This detuning corresponds to the reduction of the average longitudinal velocity due to the nonvanishing emittance and focusing by the wiggler and/or ions. Figures 1(a)–1(c) cover most of the practical range of the FEL parameters, and they can be used to obtain quick estimates of the electric-field e-folding length.

Suppose we wish to optimize the design of a FEL operating at $\lambda_r=300$ Å. We assume the source of the electron beam is a linac having fixed normalized emittance $\epsilon_n=\gamma_0\epsilon=3$ mm mrad. In the optimization, we fix the electron current $I_0=200$ A, and the electron-beam energy spread $\sigma=0.1\%$. We assume the flat wiggler to be of hybrid design with Nd-Fe-B magnet blocks and vanadium-Permendur pole tips. The peak on-axis magnetic field is related to g/λ_w (the gap-to-period ratio) by Halbach's relation[16] ($g/\lambda_w\le 0.722$):

$$B_w=3.44\exp[-5.00(g/\lambda_w)+1.54(g/\lambda_w)^2]\,. \qquad (14)$$

We fix the gap at $g=4$ mm, and consider the gain as a function of the wiggler period λ_w and the betatron wavelength $\lambda\beta$. Once λ_w is chosen, B_w is determined by Eq. (14), and γ_0 by $\lambda_r=300$ Å. In Fig. 2, we plot the e-folding length L of the electric field against the betatron wavelength $\lambda_\beta=2\pi/k_\beta$ for the following three cases: ($\lambda_w=1$ cm, $\gamma_0=439$, $B_w=0.6$ T), ($\lambda_w=2$ cm, $\gamma_0=1177$, $B_w=1.35$ T), and ($\lambda_w=3$ cm, $\gamma_0=2639$, $B_w=1.82$ T), all corresponding to radiation wavelength $\lambda_r=300$ Å. The case of $\lambda_w=2$ cm gives the shortest gain length, with the minimum corresponding to betatron wavelength $\lambda_\beta=5$ m. In this case, the natural focusing of the wiggler yields $\lambda_\beta=13$ m, so an enhancement of the gain is achieved by increasing the focusing over that produced by the wiggler itself.

Employing Eq. (11), we can discuss the scaling with energy γ_0 when we try to push to a shorter wavelength. We shall consider the wiggler parameters λ_w and K fixed, and suppose the energy γ_0 to be increased, resulting in a reduction of the radiated wavelength $\lambda_r\sim\gamma_0^{-2}$. We shall keep D [Eq. (10)] constant by increasing the current proportional to energy, $I_0\sim\gamma_0$, and maintain $k_r\epsilon$ constant by assuming the normalized emittance is reduced as $\epsilon_n\sim 1/\gamma_0$. We shall also assume the fractional energy spread σ and the betatron wavelength λ_β are held constant, and the detuning is given by Eq. (13). It then is a consequence of the scaling relation of Eq. (11) that the e-folding length L remains invariant. Note that since the wiggler's natural focusing has the dependence $\lambda_\beta\sim\gamma_0$, as the energy increases it is necessary to use external focusing of the electrons, e.g., from ions, to keep λ_β constant.

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