Soliton Solutions to the Gauged Nonlinear Schrödinger Equation on the Plane

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A gauged, nonlinear Schrödinger equation in two spatial dimensions is considered. This equation describes nonrelativistic matter interacting with Chern-Simons gauge fields. We find explicit static, self-dual solutions that satisfy the Liouville equation.

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The gauged, nonlinear Schrödinger equation for the "matter" field $\Psi(t,\mathbf{r})$ reads

$$i\hbar\partial_t\Psi(t,\mathbf{r})=\left\{-\frac{\hbar^2}{2m}\left(\nabla-\frac{ie}{\hbar c}\mathbf{A}(t,\mathbf{r})\right)^2+eA^0(t,\mathbf{r})-g\Psi^*(t,\mathbf{r})\Psi(t,\mathbf{r})\right\}\Psi(t,\mathbf{r})\,. \tag{1}$$

Here, we present the equation in its quantum-mechanical form; hence Planck's constant $\hbar=h/2\pi$ occurs. Also, m is a mass parameter, c is the velocity of light, g governs the strength of the nonlinearity, and e measures the coupling to a gauge field described by scalar (A^0) and vector ($\mathbf{A}$) potentials. This coupling is gauge invariant: A gauge transformation of the potentials

$$\mathbf{A}\to\mathbf{A}-\nabla\omega,\quad A^0\to A^0+\frac{1}{c}\partial_t\omega\,, \tag{2a}$$

accompanied by a phase redefinition of Ψ,

$$\Psi\to e^{-i(e/\hbar c)\omega}\Psi\,, \tag{2b}$$

leaves (1) unchanged.

Without gauge fields, $e=0$, and with the interpretation that Ψ is a noncommuting quantum-field operator, (1) is the Heisenberg equation of motion for a second-quantized description of point particles moving nonrelativistically in δ-function potentials with strength $-g$. Moreover, in one spatial dimension and still at $e=0$, with the interpretation that Ψ is a classical, c-number field, (1) possesses soliton solutions and is completely integrable. Indeed, understanding the soliton structure of the one-dimensional nonlinear Schrödinger equation was an important achievement in the complete integrability program for nonlinear partial differential equations[1] and in the semiclassical, nonperturbative quantization of nonlinear quantum field theories.[2]

Here, we consider the model in two spatial dimensions and with nonvanishing gauge coupling. The gauge field satisfies its own dynamical equation, where the conserved matter current J^μ acts as a source:

$$J^\mu=(c\rho,\mathbf{J})=\left(c\Psi^*\Psi,\frac{\hbar}{2mi}[\Psi^*(\mathbf{D}\Psi)-\Psi(\mathbf{D}\Psi)^*]\right),$$

$$\partial_\mu J^\mu=\partial_t\rho+\nabla\cdot\mathbf{J}=0\,. \tag{3}$$

($\mathbf{D}$ is the gauge-covariant derivative: $\mathbf{D}\Psi=[\nabla-(ie/\hbar c)\mathbf{A}]\Psi$.) However, the gauge-field equation need not be of the conventional Maxwell form: $\partial_\mu F^{\mu\nu}=(e/c)J^\nu$, $F_{\mu\nu}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu$; in planar physics the Chern-Simons term provides a possible modification. The most general, linear gauge-field equation in three-dimensional space-time reads[3]

$$\partial_\mu F^{\mu\nu}+\frac{\kappa}{2}\epsilon^{\nu\alpha\beta}F_{\alpha\beta}=\frac{e}{c}J^\nu\,. \tag{4}$$

[We use relativistic notation with the metric diag$(1,-1,-1)$ and $x^\mu=(ct,\mathbf{r})$, but the form of the modification is largely determined by current conservation.] The parameter κ, with inverse length dimensionality, controls the Chern-Simons addition and provides a cutoff at large distances greater than $1/|\kappa|$ for the gauge-invariant electric, $\mathbf{E}=-\nabla A^0-(1/c)\partial_t\mathbf{A}$, and magnetic, $\mathbf{B}=\nabla\times\mathbf{A}$, fields. [On the plane, the curl of a vector $\mathbf{V}$ is a scalar S, and the curl of a scalar is a vector: In components $S=\epsilon^{ij}\partial_iV^j$, $(\nabla\times S)^i=\epsilon^{ij}\partial_jS$.] Thus the Chern-Simons term gives rise to massive, yet gauge-invariant "electrodynamics."

The time component of (4) is the Chern-Simons modified Gauss law:

$$\nabla\cdot\mathbf{E}-\kappa B=e\rho\,. \tag{5}$$

Upon integration over the entire plane, this has the important consequence that any field configuration with charge $Q=e\int d\mathbf{r}\,\rho(t,\mathbf{r})$ also carries magnetic flux $\Phi=\int d\mathbf{r}\,B(t,\mathbf{r})$ given by

$$\Phi=-\frac{1}{\kappa}Q\,. \tag{6}$$

The first term in (5) integrates to zero owing to the long-distance damping by the "photon mass" $|\kappa|$; as mentioned above, all gauge-invariant gauge-field quantities are short range. For the same reason the spatial integral of B converges, but then it also follows that necessarily the gauge-variant vector potential $\mathbf{A}$ is long range, so that the spatial integral of $\nabla\times\mathbf{A}$ is nonzero. Thus, charged systems carry a vortexlike magnetic field.

We shall make use of an interesting truncation of the above planar, gauge-theoretic dynamics, wherein only the Chern-Simons contribution in the left-hand side of (4) is retained.[4] This "Chern-Simons electrodynamics" may be viewed as the $|\kappa|\to\infty$ limit of the topologically massive model. The truncation is physically sensible at large distances and low energies, where the lower-derivative Chern-Simons term dominates the higher-derivative Maxwell term, and the magnetic-electric relation (6) holds locally in space.

In this Letter, we present static solutions[5] to the gauged, planar nonlinear Schrödinger equation, when the nonlinear coupling g takes a particular, natural value related to the gauge coupling. Gauge-field dynamics is provided by the field-current identity, which is all that is left of (4) when the Maxwell term is dropped:

$$\frac{\kappa}{2}\epsilon^{\mu\alpha\beta}F_{\alpha\beta}=\frac{e}{c}J^{\mu}. \tag{7}$$

Equations (1) and (7) also follow from the Lagrange density,

$$\mathcal{L}=\frac{\kappa}{4}\epsilon^{\mu\alpha\beta}A_{\nu}F_{\alpha\beta}+i\hbar\Psi^*\left(\partial_t+\frac{ie}{\hbar}A^0\right)\Psi-\frac{\hbar^2}{2m}|\mathbf{D}\Psi|^2+\frac{g}{2}(\Psi^*\Psi)^2. \tag{8}$$

The gauge-field variables in (1) can be expressed, with the help of (7), in terms of the matter variables: After (7) is presented in components,

$$B=\epsilon^{ij}\partial_iA^j=-\frac{e}{\kappa}\rho, \tag{9a}$$

$$E^i=-\partial_iA^0-\frac{1}{c}\partial_tA^i=\frac{e}{c\kappa}\epsilon^{ij}J^j, \tag{9b}$$

we recognize that the potentials are given by

$$\mathbf{A}(t,\mathbf{r})=\nabla\omega(t,\mathbf{r})+\nabla\times\frac{e}{k}\int d\mathbf{r}'G(\mathbf{r}-\mathbf{r}')\rho(t,\mathbf{r}'), \tag{10a}$$

$$A^0(t,\mathbf{r})=-\frac{1}{c}\partial_t\omega(t,\mathbf{r})-\nabla\times\frac{e}{c\kappa}\int d\mathbf{r}'G(\mathbf{r}-\mathbf{r}')\mathbf{J}(t,\mathbf{r}'). \tag{10b}$$

Here G is the Green's function for the Laplacian, $\nabla^2G(\mathbf{r})=\delta(\mathbf{r})$, and ω is an arbitrary gauge which may be removed by redefining the phase of the matter field Ψ. Therefore, (1) is a self-contained, highly nonlinear equation for Ψ, which also follows from the Hamiltonian,

$$H=\int d\mathbf{r}\,\mathcal{H},\quad \mathcal{H}=\frac{\hbar^2}{2m}|\mathbf{D}\Psi|^2-\frac{g}{2}(\Psi^*\Psi)^2, \tag{11}$$

where $\mathbf{A}$ (but not A^0) occurs in $\mathcal{H}$ through the covariant derivative, and is expressed in terms of $\rho=\Psi^*\Psi$ through (10a) (with the gauge function ω absorbed in the phase of Ψ). One verifies that (1), supplemented by (10), may be presented as

$$i\hbar\partial_t\Psi(t,\mathbf{r})=\frac{\delta H}{\delta\Psi^*(t,\mathbf{r})}. \tag{12}$$

Static solutions,[5] for which the Hamiltonian evidently is stationary, obey

$$0=\left\{-\frac{\hbar^2}{2m}\mathbf{D}^2+eA^0-g(\Psi^*\Psi)\right\}\Psi. \tag{13}$$

To find them, we first observe the identity

$$|\mathbf{D}\Psi|^2=|(D_1\pm iD_2)\Psi|^2\pm\frac{m}{\hbar}\nabla\times\mathbf{J}\pm\frac{e}{\hbar c}B\Psi^*\Psi. \tag{14a}$$

Therefore, in view of (9a), the energy density $\mathcal{H}$ is

$$\mathcal{H}=\frac{\hbar^2}{2m}|(D_1\pm iD_2)\Psi|^2\pm\frac{\hbar}{2}\nabla\times\mathbf{J}-\left(\frac{g}{2}\pm\frac{e^2\hbar}{2mc\kappa}\right)(\Psi^*\Psi)^2. \tag{14b}$$

Consequently, with $g=\mp e^2\hbar/mc\kappa$, and sufficiently well-behaved fields so that the integral over all space of $\nabla\times\mathbf{J}$ vanishes, the energy is

$$H=\frac{\hbar^2}{2m}\int d\mathbf{r}\,|(D_1\pm iD_2)\Psi|^2. \tag{14c}$$

This is non-negative and vanishes—thus, attaining its minimum—when Ψ satisfies

$$D_1\Psi=\mp iD_2\Psi. \tag{15a}$$

The self-dual character of this equation is recognized when it is written as

$$\mathbf{D}\Psi=\mp i\mathbf{D}\times\Psi. \tag{15b}$$

We shall henceforth make the above choice for the strength g of the nonlinearity; as will be indicated below, this is in fact a very natural choice.

To solve (15), we note that when Ψ is decomposed into its phase and amplitude

$$\Psi=e^{i(e/\hbar c)\omega}\rho^{1/2}. \tag{16}$$

Equation (15) implies that the vector potential is given by

$$\mathbf{A}=\nabla\omega\pm\frac{\hbar c}{2e}\nabla\times\ln\rho. \tag{17}$$

With (9a), or equivalently (10a), this shows that, away from the zeros and poles of ρ, $\ln\rho$ satisfied the Liouville equation, all whose solutions are known,

$$\nabla^2 \ln\rho = \pm 2\frac{e^2}{\hbar c\kappa}\rho . \quad (18)$$

The spatial current (3), which is given by the London ansatz involving A_T, the transverse part of $\mathbf{A}$,

$$\mathbf{J} = -\frac{e}{mc}\rho \mathbf{A}_T , \quad (19a)$$

here equals, according to (17), (19a)

$$\mathbf{J} = \mp\frac{\hbar}{2m}\nabla\times\rho . \quad (19b)$$

The surface-term contribution to the energy is

$$\pm(\hbar/2)\int d\mathbf{r}\,\nabla\times\mathbf{J} = (\hbar^2/4m)\int d\mathbf{r}\,\nabla^2\rho .$$

This, indeed, vanishes for sufficiently well-behaved $\nabla\rho$.

One may explicitly verify that the second-order equation (13) is solved by the above self-dual, first-order system, which evidently provides a first integral for (13), corresponding to zero energy.

The Liouville equation possesses nonsingular solutions with non-negative ρ when the numerical constant on the right-hand side of (18) is negative. Hence, the $\pm$ sign must be chosen opposite to that of κ. The matter density that solves (18) is

$$\rho(\mathbf{r}) = \frac{4}{\alpha}\frac{|f'(z)|^2}{[1+|f(z)|^2]^2} , \quad (20)$$

$$\mathbf{r} = (r\cos\theta, r\sin\theta), \quad z = re^{i\theta} .$$

Here α is the dimensionless constant $\alpha = e^2/\hbar c|\kappa|$ and f is an arbitrary function.

When ρ vanishes, $\ln\rho$ is singular and so is $\nabla^2\ln\rho$, which according to (17) contributes to the magnetic field. Nevertheless, the complete magnetic field will remain nonsingular, because ω in (17) can be chosen to be discontinuous, so that singularities in $\nabla\times\nabla\omega$ cancel those of $\mp(\hbar c/2e)\nabla^2\ln\rho$. However, since the modulus of Ψ is $\rho^{1/2}$ and the phase is ω, discontinuities of ω must be quantized so that Ψ remains single valued when zeros of $\rho^{1/2}$ are encircled.

For example, the most general radially symmetric and nonsingular solution to the Liouville equation involves two parameters, n and r_0,

$$\rho(r) = \frac{4n^2}{\alpha r^2}\left[\left(\frac{r_0}{r}\right)^n + \left(\frac{r}{r_0}\right)^n\right]^{-2} . \quad (21)$$

This vanishes for large r, and is nonsingular at the origin for $|n| \geq 1$, but for $|n| > 1$, the vector potential acquires a singular contribution at $r=0$ from $\ln\rho$:

$$A^i = \partial_i\omega \pm \frac{\hbar c}{e}\epsilon^{ij}\frac{\hat{r}^j}{r}\left[n-1-\frac{2n}{1+(r_0/r)^{2n}}\right]$$

$$\underset{r\to 0}{\longrightarrow} \partial_i\omega \pm \frac{\hbar c}{e}\epsilon^{ij}\frac{\hat{r}^j}{r}(|n|-1) . \quad (22)$$

The singularity is removed when we chose $\omega = \pm(\hbar c/e)(|n|-1)\theta$, and so the field profile is

$$\Psi(\mathbf{r}) = e^{\pm i(|n|-1)\theta}\frac{2n}{\sqrt{\alpha}r}\left[\left(\frac{r_0}{r}\right)^n + \left(\frac{r}{r_0}\right)^n\right]^{-1} . \quad (23)$$

We now see that $|n|$ must be an integer for single-valued Ψ. For this solution the charge is

$$Q = e\int d\mathbf{r}\,\rho = \frac{4\pi|n|e}{\alpha} , \quad (24)$$

and so the flux is

$$\Phi = -\frac{\kappa}{|\kappa|}\frac{2hc}{e}|n| . \quad (25)$$

Note that the flux quantum, $e\Phi/hc$, is an even integer.

We postpone to another, longer publication further discussion of this system. Here, we conclude with two comments.

(A) Viewed as a quantum-mechanical equation for the operator field Ψ, the gauged, nonlinear Schrödinger equation provides a second-quantized description for nonrelativistic point particles interacting with a Chern-Simons gauge field, and also with a δ-function potential arising from the cubic nonlinearity in (1). However, since the magnetic field of a point source is a δ function [see (9a)], this additional interaction may alternatively be viewed as occurring because the point particles possess a magnetic moment. One finds that the special value for the nonlinear coupling that renders the system self-dual corresponds to the minimal moment of a spin-$\frac{1}{2}$ particle.

(B) Recently, there have been found topological[6] and nontopological[7] solitons in a gauged Klein-Gordon equation. When the field potential is sixth order and of special form, the solitons obey self-dual equations. The solitons in our nonlinear Schrödinger equation correspond to the nonrelativistic limit for the nontopological solitons of the Klein-Gordon model.

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