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Superpositions of Time Evolutions of a Quantum System and a Quantum Time-Translation Machine

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A method to obtain a superposition of time evolutions of a quantum system which correspond to different Hamiltonians as well as to different periods of time is derived. Its application to amplification of an effect due to the action of weak forces is considered. A quantum time-translation machine based on the same principle, utilizing the gravitational field, is also considered.

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In this Letter we introduce a new notion: a superposition of time evolutions (rather than that of states) of a quantum system. We demonstrate that there exist superpositions of several time-evolution operators U_i which, for a large class of states $|\Psi\rangle$, are effectively equal to a single but very different time-evolution operator U':

$$\sum_i c_i U_i |\Psi\rangle \cong U' |\Psi\rangle . \tag{1}$$

We show that one can select events for which the proper way to describe the time evolution of the quantum system is by this unitary operator U'. Such superpositions may lead to unusual consequences as demonstrated by two examples presented in this Letter. In the first example, we show that a suitable superposition of time evolutions due to the action of weak forces is equivalent to a time evolution resulting from the action of a strong force. This corresponds to a new method of amplification. In the second example, we show that a particular superposition of time evolutions with the same Hamiltonian but different periods of time is equivalent to a time evolution for a very different (possibly even negative) period of time. This provides an example of a new type of "time-translation machine" which is peculiar to quantum systems and has no classical analog.

In order for the superposition of time evolutions to have the form (1) the normalization of the coefficients c_i has to be

$$\sum_i c_i = 1 . \tag{2}$$

We shall use this normalization from now on.

Let us consider a quantum system S evolving for a period of time T according to a Hamiltonian which depends on a parameter a. The time evolution of a quantum system described by the state $|\Psi\rangle$ due to a superposition of evolutions with Hamiltonians which have parameters a_i is

$$\sum_i c_i U(T,a_i) |\Psi\rangle = \sum_i c_i \exp\left(-i\int_{t_0}^{t_0+T} H(a_i)dt\right) |\Psi\rangle . \tag{3}$$

We shall show that this superposition may yield, effectively, a single time evolution corresponding to a value of the parameter a' which is far out of the range of $\{a_i\}$:

$$\sum_i c_i U(T,a_i) |\Psi\rangle \cong U(T,a') |\Psi\rangle . \tag{4}$$

The superposition (3) can be obtained in the following way. Let the Hamiltonian be a function of a conserved quantum variable A related to an external system S_e, and $\{a_i\}$ be the eigenvalues of the corresponding operator A. Let the initial state (not normalized) of the external system S_e at time T_0 be $|\Phi_1\rangle = \sum_i c_i |a_i\rangle$. This will lead to the following time evolution of the relevant parts of our combined system:

$$\sum_i c_i \exp\left(-i\int_{t_0}^{t_0+T} H(a_i)dt\right) |\Psi\rangle |a_i\rangle . \tag{5}$$

This is not yet a superposition of time evolutions for the system S itself. Instead, we have different evolutions correlated to different values of A. To obtain the desired superposition described by Eq. (3) we have to perform a projection on a particular state of the external system S_e. Indeed, if at the time t_0+T an appropriate measurement is performed on the external system and its resulting state is found to be $|\Phi_2\rangle=(1/\sqrt{N})\sum_i|a_i\rangle$, then the superposition (3) describes the corresponding state of our system. In fact, the superposition of the time evolutions (3) can be obtained also by preselecting the initial state of the external system $|\Phi_1\rangle=\sum_i\alpha_i|a_i\rangle$ and postselecting the state $|\Phi_2\rangle=\sum_i\beta_i|a_i\rangle$ provided

$$\frac{\alpha_i\beta_i^*}{\sum_i\alpha_i\beta_i^*}=c_i\,. \tag{6}$$

As an example, consider a system which experiences a force in the x direction whose magnitude depends on a quantum variable A:

$$H=-xA\,. \tag{7}$$

Then the superposition (3) yields at time t_0+T the following wave function in the p representation:

$$\Psi(p,t_0+T;\{c_i\})=\sum_i c_i\Psi(p-a_iT,t_0)\,. \tag{8}$$

As we shall show, there are sets $\{c_i\}$ and $\{a_i\}$ such that for a wide range of the initial wave functions $\Psi(p,t_0)$ the superposition (8) is equivalent to an evolution under a single Hamiltonian with a parameter a' which lies far outside the range of $\{a_i\}$:

$$\sum_i c_i\Psi(p-a_iT)\cong\Psi(p-a'T)\,, \tag{9}$$

where

$$a'\equiv\sum_i c_ia_i\,. \tag{9a}$$

If the superposition were obtained by preselecting the state $|\Phi_1\rangle$ and postselecting the state $|\Phi_2\rangle$ of the external system, then the c_i are given by

$$c_i=\frac{\langle\Phi_2|a_i\rangle\langle a_i|\Phi_1\rangle}{\langle\Phi_2|\Phi_1\rangle}\,, \tag{10}$$

and a' is given by

$$a'=\frac{\langle\Phi_2|A|\Phi_1\rangle}{\langle\Phi_2|\Phi_1\rangle}\,. \tag{11}$$

For sufficiently small T Eq. (9) is correct, in fact, for any choice of $\{c_i\}$ and $\{a_i\}$. Indeed, the first two terms of the Taylor expansion of the right-hand and left-hand sides of (9) are identical, and for small T the higher terms may be neglected. Thus, the superposition of forces a_i acting for a short period of time is equivalent to a force a' acting for the same period of time. The meaning of "short" here is that during this time the state of the system does not change significantly. Therefore, even if the effective value of the force a' is much bigger than any of the a_i, its effect on the system is so small that it can be detected only if we have a large ensemble of identical systems.

More interesting is the situation when Eq. (9) describes correctly the wave function of the system for a finite period of time. A sufficient condition for this to happen is that the higher-order terms of the Taylor expansion of both sides of Eq. (9) are approximately equal:

$$\sum_i c_ia_i^k\cong(a')^k\quad\text{for all } k<K\,. \tag{12}$$

We shall describe now one of the procedures to construct the sets $\{c_i\}$ and $\{a_i\}$ which fulfill this requirement.

Consider a quantum system with a variable A which has a discrete set of eigenvalues $\{a_i\}$ and quantum states $|\Phi_1\rangle$ and $|\Phi_2\rangle$ such that a' [see Eq. (11)] is far outside of the range of $\{a_i\}$. In general, the requirement (12) with coefficients c_i given by Eq. (10) will not be fulfilled for $k\geq 2$. Now we shall redefine the variable, the states, and the system S_e itself. The new system will be an ensemble of N systems identical to the one described above, the states describing this ensemble will be given by the products of the above states, and the variable A will be chosen to be the average of A_n, i.e.,

$$|\Phi_1\rangle=\prod_n|\Phi_1\rangle_n\,,\qquad |\Phi_2\rangle=\prod_n|\Phi_2\rangle_n\,,\qquad A\equiv\frac{1}{N}\sum_n A_n\,. \tag{13}$$

This yields a new set of eigenvalues $\{a_i\}$ and a new set of coefficients $\{c_i\}$ but the value a' and the range of the eigenvalues of A remain unchanged. The corrections to Eq. (12) for this case, however, are negligible as long as $K\ll N$. Indeed, the left-hand side of Eq. (12) can be shown to be

$$\frac{(\prod_n\langle\Phi_2|_n)[(1/N)\sum_nA_n]^k(\prod_n|\Phi_1\rangle_n)}{(\prod_n\langle\Phi_2|_n)(\prod_n|\Phi_1\rangle_n)}\,.$$

It differs from the right-hand side of Eq. (12) only because of the terms in the polynomial expansion of $(\sum_nA_n)^k$ in which at least one A_n is repeated. But, if $k\ll N$, the contribution arising from these terms can be neglected. In fact, Eq. (12) here represents the fact that fluctuations for a large ensemble are negligible.

An ensemble of N spin-$\frac{1}{2}$ particles with identical states can be replaced, in our procedure, by a spin-$N/2$ particle in a state for which a given spin component has its maximal possible value N (in units of $\frac{1}{2}\hbar$). We can, therefore, illustrate our method using a spin-$N/2$ particle as our system. In this case there is no external system: The spatial variables of the particle play the role of the variables of the external system, while its spin component in the x direction S_x plays the role of the variable A. Consider then a spin-$N/2$ particle which has been prepared at time t_0 in the state $|\Phi_1\rangle=|S_\xi=N\rangle$ and at time t_0+T has been postselected in the state $|\Phi_2\rangle = |S_\eta = N\rangle$. The directions $\hat{\xi}$, $\hat{\eta}$, and $\hat{x}$ are lying in one plane, and $\hat{\xi}$ and $\hat{\eta}$ create the same angle θ with the x axis. The force on the particle is assumed to be proportional to the value of the x component of its spin (Stern-Gerlach device): $A \equiv S_x/N$. Thus, the eigenvalues of A are

$$a_i = \frac{N-2i}{N}, \quad i=0,1,\ldots,N, \tag{14}$$

the coefficients c_i are given by Eq. (10),

$$c_i = \frac{\langle S_\eta = N | S_x = N-2i\rangle\langle S_x = N-2i | S_\xi = N\rangle}{\langle S_\eta = N | S_\xi = N\rangle} = \left(\frac{\cos^2(\theta/2)}{\cos\theta}\right)^N \left(-\tan^2\frac{\theta}{2}\right)^i \frac{N!}{i!(N-i)!}, \tag{15}$$

and the "effective" value of the force a' is [see Eq. (11)]

$$a' = \frac{\langle S_\eta = N | (S_x/N) | S_\xi = N\rangle}{\langle S_\eta = N | S_\xi = N\rangle} = \frac{1}{\cos\theta}. \tag{16}$$

When the angle θ is close to $\frac{1}{2}\pi$ the effective force becomes very large. Thus, *the effect of superposing time evolutions corresponding to weak forces* ($|a_i| \leq 1$) *is equivalent to the effect of a single strong force* ($a' = 1/\cos\theta$). This is a form of amplification which is peculiar to quantum mechanics; it has no classical analog since it is due to quantum interferences. We hope that this amplification scheme can have practically useful applications. A realistic proposal for an optical analog of the amplification in the Stern-Gerlach experiment[1] was recently suggested.[2] For discussion see also Refs. 3 and 4.

For N large enough the requirement (12) is fulfilled for a given K with any desired precision. Still, in order to have our effect for a large period of a time T we have to make some restrictions on the spatial wave function of our particles. The wave function in the p representation $\Psi(p)$ has to be such that the higher orders in the Taylor expansion of both sides of Eq. (9) can be neglected. The requirement on $\Psi(p)$ became weaker for larger N. Or, for a given $\Psi(p)$, the maximal possible amplification increases with N. [It is proportional, approximately, to $\sqrt{N}$, see Ref. 3, Eq. (30).] An example of equality (9) for a Gaussian $\Psi(p) = e^{-p^2/4(0.8)^2}$, $T=1$, $N=15$, and $\theta = 70°$ is shown in Fig. 1.[5] In order to obtain an appropriate $\Psi(p)$ in practice we can pass our particle through a slit. It is interesting (in particular, for practical realization of this method) that the slit can be placed in any stage of the experiment: before the first filter which preselects $|\Phi_1\rangle$, before or after the interaction, or even after the particle has passed through all filters.

In the example presented above we considered a superposition of time evolutions corresponding to different Hamiltonians all acting during the same period of time. We shall show now how it is possible to obtain a superposition of time evolutions corresponding to the same Hamiltonian acting during different periods of time and how it can be used for constructing a quantum time-translation machine.

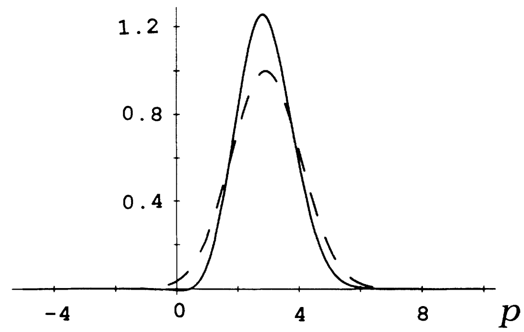

FIG. 1. Superposition of Gaussians centered between -1 and 1 yields, approximately, a Gaussian centered at the value 3. The solid line represents the left-hand side, and the dashed line the right-hand side of Eq. (9) for $\Psi(p) = e^{-p^2/4(0.8)^2}$, $T=1$; a_i is given by Eq. (14), and c_i is given by Eq. (15) with $N=15$, $\theta = 70°$.

Again, the basis of the method is that the superposition of the time evolutions during the periods of time T_i is, effectively, the time evolution during the time $T' = \sum_i c_i T_i$ which may be very different from the range of $\{T_i\}$:

$$\sum_i c_i U(T_i) |\Psi\rangle \cong U(T') |\Psi\rangle. \tag{17}$$

This equality holds provided the following requirements are fulfilled. Similarly to Eq. (12) the requirements for $\{T_i\}$ and $\{c_i\}$ are

$$\sum_i c_i T_i^k \cong (T')^k \quad \text{for all } k < K. \tag{18}$$

For an isolated system the time-evolution operator is $U(T) = \exp(-iHT)$, and the requirement on the Hamiltonian and the initial state $|\Psi\rangle$ is that the energy distribution diminishes fast enough for large energies. Thus, for a large class of Hamiltonians and initial states $|\Psi\rangle$ the superposition (17) with a particular choice of $\{c_i\}$ and $\{T_i\}$ amounts, effectively, to a single time evolution toward the past (for negative T') or toward remote future (for large positive T').

One of the ways to achieve superpositions of this kind is to send our system on a journey in a rocket whose velocity, and therefore whose relativistic time delay, is correlated to a quantum variable. Another procedure is to surround our system with a massive spherical shell of radius R_0 and then to build a mechanism which will change the radius of the shell to the value R at time t_0 and which will bring it back to R_0 at the later time t_0+T. The mechanism is such that the radius R depends on an external quantum variable. Preselection measurement before time t_0 and postselection measurement after time t_0+T performed on the external system will produce a superposition of time evolutions corresponding to different periods of the proper time of our system.

Indeed, the time period T which is defined for an external observer (say, at infinity) will correspond to different periods of the proper time T_i for the system inside the shell. The Newtonian potential inside the shell yields the following element of the space-time metric: $g_{00}=1-2GM/c^2R$, where M and R are the mass and the radius of the shell, respectively. Therefore, the periods of the proper time T_i corresponding to radii R_i are

$$T_i = T\left(1-\frac{2GM}{c^2R_i}\right)^{1/2}. \tag{19}$$

Let us prepare an external quantum system before the time t_0 in the state $\sum_i c_i|i\rangle$ and let us assume that after the time t_0+T it was found in the state $(1/\sqrt{N})\sum_i|i\rangle$. Then, the time evolution of the relevant parts of the combined system (our system, shell, and an external quantum system), shown at the times before t_0, before t_0+T, and after t_0+T, is

$$\left(\sum_i c_i|i\rangle\right)|R_0\rangle|\Psi\rangle \to \sum_i c_i|i\rangle|R_i\rangle e^{-iHT_i}|\Psi\rangle \to \left(\frac{1}{\sqrt{N}}\sum_i|i\rangle\right)|R_0\rangle \times\left(\sum_i c_i e^{-iHT_i}\right)|\Psi\rangle. \tag{20}$$

Thus, after the final measurement is performed on the external system, the system inside the shell is in a pure state corresponding to the superposition of time evolutions. If $\{c_i\}$ and $\{T_i\}$ fulfill the requirements (18), and the energy distribution decreases fast enough for large energies then, effectively, our system moves [see Eq. (17)] to the time t_0+T' which may differ significantly from the time of the external observer t_0+T.

Apart from the technical difficulties of building this type of a time-translation machine there is another problem. This machine will succeed to work only very rarely. We need a particular outcome of the postselection measurement and for any significantly long journey in time the probability to obtain this outcome is extremely small. While we believe that the amplification effect described above can find its way toward practical applications, the gravitational time-translation machine is a gedanken experiment.

We have demonstrated that it is possible to prepare a superposition of time evolutions U_i which, for certain states $|\Psi\rangle$, is effectively equal to a very different time evolution U'. In the two examples which we have discussed, Eq. (1) is not correct for all possible states of the system. In the first example, the spatial wave function has to decrease fast enough for a large x, and in the example of a time-translation machine, the energy distribution has to decrease fast enough for large energies. However, for certain systems Eq. (1) holds for *all* states $|\Psi\rangle$ and then it can be replaced by the operator equation

$$\sum_i c_i U_i \cong U'. \tag{21}$$

Indeed, this is the situation for a system confined in a box and described by the Hamiltonian (7). It is also the case of our time-translation machine working on systems described by bounded Hamiltonians.

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[5]The probability to obtain this situation, i.e., to get an outcome of the postselection measurement $S_\eta=N$, is approximately $(\cos^2\theta)^N \cong 10^{-7}$.