Low-Temperature Phase of the Ising Spin Glass on a Hypercubic Lattice

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We consider the free energy, averaged over disorder using the replica method, of an Ising spin glass on a d-dimensional hypercubic lattice. We demonstrate that the free energy can be expanded in powers of $1/d$, and that the zeroth-order ($d=\infty$) result recovers thermodynamics identical to that of the Sherrington-Kirpatrick model. We explicitly solve the model near (and below) its spin-glass critical temperature to order $1/d^2$, and find an enhancement of replica-symmetry-breaking effects as the dimension is decreased.

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The replica method was introduced into spin-glass physics in 1975 (Ref. 1) in an attempt to construct a sensible "mean-field" theory. Parisi's replica-symmetry-breaking solution[2] of the infinite-ranged SK (Sherrington-Kirpatrick) model[3] is now generally accepted as correct, and the replica method has in the last few years been successfully used to study many other disordered but infinite-ranged problems, including optimization problems[4] and the Hopfield neural network model.[5] Despite this enormous theoretical progress, we nevertheless still face the question of the relevance of the Parisi mean-field solution to the real short-ranged three-dimensional spin glasses[6] which are studied experimentally. In particular, one might worry that the complex features of Parisi's solution associated with replica symmetry breaking are only pathologies associated with the infinite-ranged nature of the SK model, and would not survive in a more realistic short-ranged model. In this Letter, we report on a direct approach to the short-ranged spin glass on a d-dimensional hypercubic lattice, and show that replica-symmetry-breaking effects are actually enhanced in comparison to the SK model, at least to leading orders in $1/d$.

One of the most important predictions of the mean-field theory is the existence at low temperatures of an infinite number of pure equilibrium states, unrelated to each other by any obvious symmetry, and organized in a hierarchical structure. Although some important experimental facts are qualitatively well accounted for by this picture (beginning with the existence of a transition in a magnetic field), a full quantitative comparison has been impossible so far. The technical reason is that the usual loop expansion around the mean-field theory is very complicated and has resisted a complete understanding.[7] On the other hand, it is also possible to build a phenomenological droplet model of the low-temperature phase, assuming that there exist in zero field only two pure states, related to each other by the global symmetry of reversal of all spins.[8] This picture can also account for some experimental observations, but exhibits some fundamental differences with mean-field theory including the prediction of no real transition in a field.

In this Letter, we consider the Hamiltonian of an Edwards-Anderson Ising spin glass on a d-dimensional hypercubic lattice:

$$H=-\sum_{(ij)} J_{ij} S_i S_j \,, \tag{1}$$

with N Ising spins S_i, and quenched random exchange couplings J_{ij} between nearest-neighbor spins on the lattice. The J_{ij}'s are chosen from a Gaussian probability distribution with zero mean and variance $1/2d$.

To perform an average over the disorder, we introduce at each site i, n replicas S_i^a, $a=1,\dots,n$, of the original spin variables. We recover the desired quenched average of the free energy in the $n\to 0$ limit:[2]

$$-\beta\bar{F}(\beta)=\lim_{n\to 0}\frac{1}{n}\left[\ln \operatorname{Tr}\exp\left(\frac{\beta^2}{4d}\sum_{(ij)}\sum_{ab} S_i^a S_i^b S_j^a S_j^b\right)\right], \tag{2}$$

where β is the inverse temperature. By expanding around $\beta=0$, one can easily recover from this expression the ordinary high-temperature expansion of the d-dimensional spin glass.[9,10] To calculate quantities in the low-temperature phase, we introduce Lagrange multipliers which fix the appropriate order parameters, in our case the overlap between two replicas:

$$q_i^{ab}=\langle S_i^a S_i^b\rangle \quad (1\le a<b\le n)\,, \tag{3}$$

where the angular brackets indicate a thermal average. Thus we consider the *Gibbs* free energy,

$$-\beta\bar{G}(q_i^{ab},\beta)=\lim_{n\to 0}\left[\frac{1}{n}\ln \operatorname{Tr}\exp\left(\frac{\alpha}{d}\sum_{(ij)}\sum_{ab} S_i^a S_i^b S_j^a S_j^b+\sum_{i,a<b}\lambda_i^{ab}(\alpha)(S_i^a S_i^b-q_i^{ab})\right)\right], \tag{4}$$

where $\alpha = \beta^2/4$ and the $\lambda_i^{ab}(\alpha)$ are α-dependent Lagrange multipliers which enforce the constraint (3). The q_i^{ab} order parameters can actually be taken to be uniform, as the system is translationally invariant after averaging over disorder.

We will now expand $\beta\bar{G}$ in a power series in α, and then analyze the resulting free energy as a function of q_i^{ab}. A similar procedure using a high-temperature expansion at a fixed magnetization was developed long ago to analyze the Ising ferromagnet in its low-temperature phase.[11] That ferromagnetic expansion can also be converted into a $1/d$ expansion, although other possible ways to organize the expansion around mean-field theory may be considered as well.[12] For the *unreplicated* spin glass, with Lagrange multipliers which fix the site-dependent magnetization, such a procedure yields the Thouless-Anderson-Palmer (TAP) equations[13] at second order[14] in the "high-temperature" expansion and corrections to those equations at higher order.[12]

In our problem, to obtain the free energy to a given order in $1/d$, we need only to compute a finite number of terms in the α expansion. In infinite dimensions, the only contributions are from terms of order α^0 and α^1, which give back the SK free energy. To first order in $1/d$, we only need to add contributions up to order α^2. To calculate to order $1/d^2$ (the first order at which we can see the difference from the Bethe lattice[15] coming from loops) we need to compute all terms in the free energy to order α^4. We can also easily obtain results for the Bethe lattice (where the coordination number $z=2d$) by simply dropping the loop contributions, and since some of the results for the Bethe lattice are also new, we report them as well.[16] We have computed all terms to order α^4 and $1/d^2$, but to give the reader an idea of the form of the expressions,[17] we explicitly write the free energy in zero field to order α^2 or $1/d$:

$$-\frac{\beta\bar{G}}{N} = \lim_{n\to 0}\frac{1}{n}\left[\ln \mathrm{Tr}\exp\left(\sum_{a<b}\lambda_{ab}(0)S_aS_b\right) - \sum_{a<b}\lambda_{ab}(0)q_{ab} + \frac{\beta^2}{4}\sum_{ab}(q_{ab})^2 + \frac{\beta^4}{32d}\sum_{abcd}(\langle S_aS_bS_cS_d\rangle - q_{ab}q_{cd})^2\right], \tag{5}$$

where the last sum includes potentially repeated indices.

We have chosen to work near T_c, where Parisi's replica-symmetry-breaking scheme can readily be implemented completely, and where the terms like $\lambda_{ab}(0)$ or $\langle S_aS_bS_cS_d\rangle$ (with a, b, c, and d unequal) can be expanded in powers of q_{ab}. The Gibbs free energy to order q^4 can be written

$$-\frac{\beta\bar{G}}{N} = A_0 + \lim_{n\to 0}\frac{1}{n}\left[A_2\sum_{a\neq b}q_{ab}^2 + A_3\sum_{abc\neq}q_{ab}q_{bc}q_{ca} + A_4\sum_{a\neq b}q_{ab}^4 + B_4\sum_{abcd\neq}q_{ab}q_{bc}q_{cd}q_{da} + C_4\sum_{a\neq b,c\neq b}q_{ab}^2q_{bc}^2 + \frac{\beta^2h^2}{2}\left(n+\sum_{a\neq b}q_{ab}\right)\right], \tag{6}$$

where the A's, B's, and C's are functions of β and d. We have introduced a uniform magnetic field h. Near T_c, $h^2 \sim q^3$.

The form of the free energy that we have written in Eq. (6) is in fact the most general Landau free energy to order q^4 that is consistent with the symmetries of our model. One can use this free energy to derive results that hold generally for any Ising spin-glass model. We have implemented the full Parisi replica-symmetry-breaking scheme for arbitrary A, B, and C coefficients and calculated all the coefficients in explicit α and $1/d$ expansions. The technical advantage of organizing the calculation this way is that we can break replica symmetry once for the general model, and then simply calculate successive terms to the desired order for the A, B, and C coefficients.

In Parisi's scheme, the order parameter q_{ab} ultimately becomes a function $q(x)$ where $0 \le x \le 1$.[2] The replica-symmetric solution is simply $q(x) \sim$ const. In general, however, one finds that for small $q(x)$ in a magnetic field, $q(x)$ has the shape shown in Fig. 1, with two plateaus connected by a line with slope σ. For the free energy given in Eq. (6), one finds

$$\sigma = \frac{A_3}{4(A_4+B_4)} + \cdots, \tag{7}$$

$$q(0)^3 = \frac{\beta^2h^2}{16(A_4+B_4)} + \cdots, \tag{8}$$

and

$$q(1) = \frac{A_2}{3A_3} + \frac{2A_2^2(3A_4+8B_4-C_4)}{27A_3^3} + \cdots, \tag{9}$$

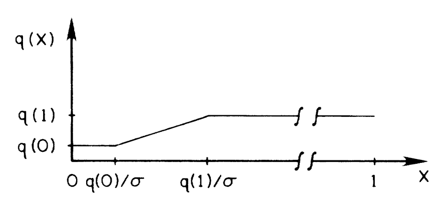

FIG. 1. The order-parameter function $q(x)$ for an Ising spin glass near T_c.

TABLE I. The Landau coefficients of the Ising spin glass expanded to $O(\beta^8)$.

$$A_0=\ln 2+\frac{\beta^2}{4}-\frac{\beta^4}{16}+\frac{\beta^6}{24d^2}-\frac{\beta^8}{384d^3}[6w(d-1)+17]+\cdots$$
$$A_2=-\frac{1}{4}+\frac{\beta^2}{4}-\frac{3\beta^4}{8d}+\frac{2\beta^6}{3d^2}-\frac{\beta^8}{32d^3}[14w(d-1)+43]+\cdots$$
$$A_3=\frac{1}{6}-\frac{\beta^4}{4d}+\frac{5\beta^6}{6d^2}-\frac{\beta^8}{48d^3}[30w(d-1)+109]+\cdots$$
$$A_4=\frac{5}{24}+\frac{\beta^4}{16d}-\frac{23\beta^6}{24d^2}+\frac{\beta^8}{192d^3}[354w(d-1)+24d+931]+\cdots$$
$$B_4=-\frac{1}{8}+\frac{\beta^4}{16d}-\frac{\beta^6}{8d^2}+\frac{\beta^8}{192d^3}[18w(d-1)+24d+91]+\cdots$$
$$C_4=-\frac{1}{4}-\frac{\beta^4}{8d}+\frac{3\beta^6}{2d^2}-\frac{\beta^8}{96d^3}[282w(d-1)+24d+721]+\cdots$$

where the $\cdots$ represents higher-order terms in $t \equiv (T_c-T)/T_c$. [A_2 is $O(t)$.]

We have performed the Almeida-Thouless[18] (AT) general stability analysis as well as the Thouless-Almeida-Kosterlitz[2] (TAK) longitudinal stability analysis, and found that they both give an AT line at

$$h^2=16T_c^2(A_4+B_4)(A_2/3A_3)^3 \tag{10}$$

which also coincides with the criterion that $q(0)=q(1)$. Note that the replica-symmetric solution is always stable if $A_4+B_4<0$, so A_4+B_4 gives a convenient measure of replica-symmetry-breaking effects.

In Table I we give our results for the A, B, and C coefficients to order α^4, where $w=1$ for the hypercubic lattice and $w=0$ for the Bethe lattice.[17] We can organize all our results into $1/d$ expansions, although, of course, other possible organizations also exist.

T_c in zero field is the point at which A_2 vanishes. Using our result from Table I, we find

$$\beta_c=1+\frac{3}{4d}+\frac{61+84w}{96d^2}+\cdots \tag{11}$$

in agreement with previous results.[10,19] Near T_c and to order $1/d^2$, we find

$$A_4+B_4=\frac{1}{12}+\frac{1}{8d}+\frac{93w-22}{48d^2}+\cdots. \tag{12}$$

Note that A_4+B_4 is positive for infinite dimensions, and *increases* as d decreases—even more strongly for the hypercubic lattice than for the Bethe lattice. We find for the order-parameter function (to leading order in t)

$$\sigma=\frac{1}{2}-\frac{3}{2d}+\frac{21-54w}{4d^2}+\cdots, \tag{13}$$

$$q(0)^3=h^2\left[\frac{3}{4}+\frac{44-129w}{8d^2}+\cdots\right], \tag{14}$$

and

$$q(1)=t\left[1+\frac{2-9w}{6d^2}+\cdots\right]. \tag{15}$$

The AT line is given by

$$\frac{h^2}{t^3}=\frac{4}{3}+\frac{204w-76}{9d^2}+\cdots, \tag{16}$$

which is in agreement with previous results[19] for the Bethe lattice. Again we see that contrary to widespread suspicion, the effect of loops is to *increase* the breaking of replica symmetry. For instance, the $1/d^2$ term indicates that the AT line moves upwards compared to the Bethe lattice. Given the fact that there always exists an AT line for the Bethe lattice with the boundary conditions that we consider,[16,19] our results lend support to the notion that an AT line always exists for the spin glass on a finite-dimensional hypercubic lattice above its lower critical dimension.[20]

The present series are rather short, and it is difficult to reliably make detailed quantitative comparisons with simulations or experiments. Previously obtained longer $1/d$ series for T_c (Ref. 10) are quantitatively consistent with results obtained from simulations[21] in three dimensions, so one can expect that longer series would also be very useful for general properties of the low-temperature phase. Nevertheless, one can still make some qualitative statements about the three-dimensional case. First, our results are possibly consistent with a critical dimension greater than three at which the *nature* of replica symmetry breaking changes—in particular, our series indicate that σ might vanish at some critical dimension, which would presumably imply a change in the zero-field linear behavior of $q(x)$ near $x=0$.[22] Second, and most importantly, the first terms of the $1/d$ expansion suggest that the main qualitative features of the mean-field solution, including the transition in a field, will survive in three dimensions. Recent simulations in three[23] and four[24] dimensions lend further credence to this view.

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