Statistics of Ballistic Agglomeration

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We consider a "sticky gas" in which collisions between spherical particles are perfectly inelastic. Thus the two colliding particles conserve mass and momentum, but merge to form a single more massive sphere. A scaling argument suggests that the average mass of a particle grows as $t^{2D/(2+D)}$, where D is the spatial dimension. In the case $D=1$ this result is confirmed by numerical simulation.

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Consider an ensemble of spheres with an average radius r_0 at $t=0$. At this initial time the spheres are distributed at random in a D-dimensional space with average separation a_0 and an average mass m_0. The velocities have an rms value u_0. We suppose that spheres move freely until they collide, at which instant they merge forming larger spheres. Mass and momentum are conserved so that if the mass density of the spheres is uniform and constant then $m(t)\sim r^D(t)$. The result of this *ballistic agglomeration* is ever larger particles moving ever more slowly and separated by increasingly greater distances. How do $m(t)$, $u(t)$, etc., scale with time? Does a universal distribution of masses develop as $t\rightarrow\infty$?

This problem provides a simple test case for scaling arguments which are being used in fluid mechanics to analyze the statistics of the merger of coherent structures such as vortices[1,2] and thermal plumes.[3,4] It is also an elementary analog of astrophysical models of the accumulation of cosmic dust into planetesimals and thence into planets.[5] As posed above it is considerably simpler than either of these applications because the particles only interact by collision. In the astrophysical context, gravitational forces are important in the later stages when the particles become large. And in the hydrodynamic case, vortices move by mutual advection. Because it avoids the complications of Brownian motion, it is simpler than Smoluchowski's discussion of colloidal coagulation.[6,7] Finally, it differs from diffusion-limited aggregation[8] because we suppose that after collisions the aggregates rapidly reorganize themselves into a compact form with fixed density. Nonetheless, this idealized model isolates some of the physics involved in these more complicated problems.

We begin our discussion with a scaling argument which shows that in D dimensions the average mass of a particle increases like $t^{2D/(D+2)}$. Initially the "collision time" τ_0 is the ratio of the volume per particle to the rate at which a sphere "sweeps out" volume:

$$\tau_0=a_0^D/u_0r_0^{D-1}\,. \tag{1}$$

This estimate is only valid at $t=0$, when a_0, r_0, and u_0 are still the characteristic scales of the ensemble.

Now define $N(t/\tau_0)$ as the expected number of initial spheres which have been condensed into a single sphere at t. This is a dimensionless function with a dimensionless argument and clearly $N(0)=1$. At $t_1\equiv s\tau_0$ we have a new initial condition with scales

$$\begin{aligned} &a_1=N_1^{1/D}a_0\,,\quad m_1=N_1m_0,\quad r_1=N_1^{1/D}r_0\,,\\ &u_1=u_0/N_1^{1/2},\quad \tau_1=N_1^{(D+2)/2D}\tau_0\,, \end{aligned} \tag{2}$$

where $N_1\equiv N(s)$. The scaling for u_1 in (2) follows from the central-limit theorem—the momentum is the sum of N_1 random vectors with typical length m_0u_0. Thus, the expected size of the resultant random momentum is $N_1^{1/2}m_0u_0$. Dividing this by the expected mass we obtain $u_1=u_0/N_1^{1/2}$.

Now consider $t_2=s\tau_1=sN_1^{1/\xi}\tau_0$, where $\xi\equiv2D/(D+2)$. At t_2 we can calculate the expected size of a particle using either $t=0$ or $t=t_1=s\tau_0$ as an initial condition. In the first case, $N(t_2/\tau)=N(sN_1^{1/\xi})$ initial particles have agglomerated into a single sphere at t_2. In the second case, each aggregate at t_2 is composed of $N(t_2/\tau_1)=N_1$ particles from t_1, and each of these contains, on average, N_1 initial particles. Equating these two different estimates,

$$N(sN(s)^{1/\xi})=N(s)^2 \text{ or } N(s)\sim s^{\xi}\,. \tag{3}$$

Thus, if $D=1$, the expected mass of a particle grows as $t^{2/3}$ and if $D=3$ the expected mass increases as $t^{6/5}$.

Before we compare these results with simulations in one dimension we introduce the density function. With $D=1$, $F(m,u,t)dm\,du$ is the number of particles per unit length with velocity in the interval $(u,u+du)$ and mass in the interval $(m,m+dm)$. Thus the number of particles per unit length and the mass per unit length are

$$\begin{aligned} &P(t)=\int_0^\infty dm\int_{-\infty}^\infty du\,F(m,u,t)\,,\\ &\frac{m_0}{a_0}=\int_0^\infty dm\int_{-\infty}^\infty du\,mF(m,u,t)\,. \end{aligned} \tag{4}$$

The scaling argument suggests that this density function evolves into a similarity form,

$$F(m,u,t) = \mathcal{F}(\mu,\nu)/m_0 u_0^2 t\,, \tag{5}$$

where

$$\mu \equiv \frac{m}{m_0}\left[\frac{tu_0}{a_0}\right]^{-2/3}, \quad \nu \equiv \frac{u}{u_0}\left[\frac{tu_0}{a_0}\right]^{1/3}.$$

In these variables (4) becomes

$$P(t) = \frac{1}{a_0}\left[\frac{a_0}{tu_0}\right]^{2/3}\int_0^\infty d\mu \int_{-\infty}^{\infty} d\nu\, \mathcal{F}(\mu,\nu)\,,$$
$$1 = \int_0^\infty d\mu \int_{-\infty}^{\infty} d\nu\, \mu \mathcal{F}(\mu,\nu)\,. \tag{6}$$

The average mass of a particle is $\langle m\rangle = m_0/a_0 P(t) \sim t^{2/3}$ and the particle-weighted average of any function, $g(m,u)$, is

$$\langle g(m,u)\rangle \equiv \frac{\int_0^\infty dm \int_{-\infty}^{\infty} du\, g(m,u) F(m,u,t)}{\int_0^\infty dm \int_{-\infty}^{\infty} du\, F(m,u,t)}\,, \tag{7}$$

so that the similarity theory predicts $\sigma_m^2 \equiv \langle (m - \langle m\rangle)^2\rangle \sim t^{4/3}$ and $u_{\text{rms}} \equiv \langle u^2\rangle^{1/2} \sim t^{-1/3}$.

Figure 1 shows the results of numerical simulation of the one-dimensional case. Initially there were 100000 particles with $m=1$ equally spaced in $0 < x < 1$. The initial velocities were uniformly distributed in the interval $(-\frac{1}{2}, +\frac{1}{2})$. The particles are unconstrained at $x=0$ and $x=1$ so they can "escape." However, a substantial number of collisions occur before this becomes a problem. Figure 1 shows that after a transient subsides the average mass increases as $t^{2/3}$, in accord with the scaling argument. As support for the similarity solution in Eq. (4) we also show the $t^{2/3}$ growth of σ_m. A similar analysis confirms that the rms velocity is decreasing as $t^{-1/3}$ in this scaling regime.

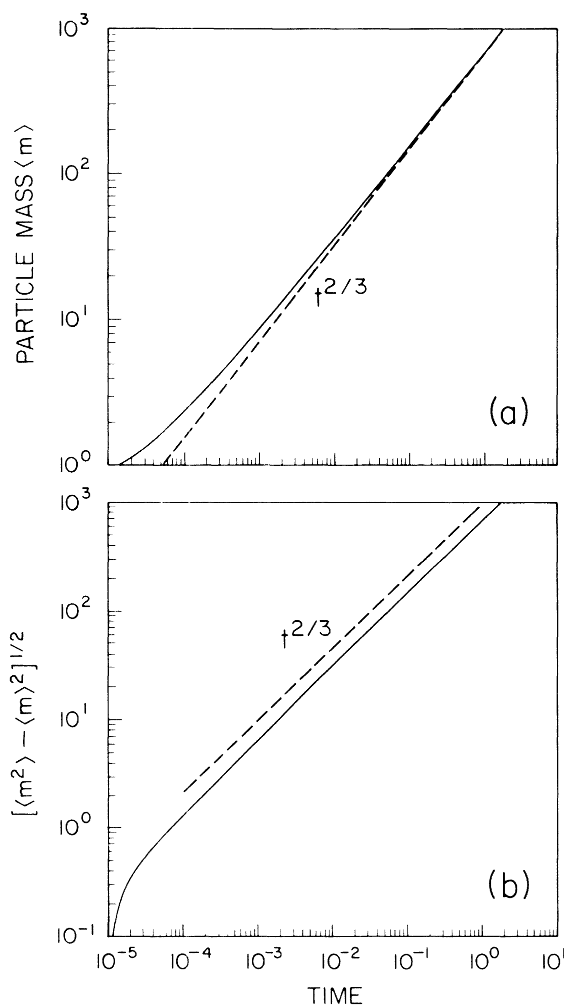

FIG. 1. (a) Average mass $\langle m\rangle$ and (b) standard deviation σ_m in a simulation starting with 100000 particles. The ratio $\langle m\rangle/\sigma_m$ approaches 1 which is consistent with the exponential density in Eq. (8).

In a second simulation, again with 100000 particles at $t=0$, the initial velocities were $\pm\frac{1}{2}$ with equal probability, while the initial positions were uniformly distributed in $0 < x < 1$. Again, after a transient, there is a scaling regime in which the average mass increases as $t^{2/3}$ and the rms speed decreases as $t^{-1/3}$. We also performed several smaller simulations with 10000 particles using a variety of initial distributions of velocity and position. All of these calculations support the hypothesis that there is a scaling regime as in Eq. (5) and the density function $F(m,u,t)$ does not depend on the initial conditions.

A detailed examination of the velocity and mass statistics produced by these simulations suggests that in the scaling regime the density function is

$$F(m,u,t) = \frac{P\exp[-m/\langle m\rangle - u^2/2u_{\text{rms}}^2]}{\langle m\rangle (2\pi u_{\text{rms}}^2)^{1/2}}\,. \tag{8}$$

A proof of this conjecture, if true, is not easy. It is possible to write down a Boltzmann equation allowing for the complication of coalescing particles. Direct substitution of (8) into this equation fails. In fact, the *stosszahlansatz* used to construct the Boltzmann equation is invalid because of strong position-velocity correlations which develop in the similarity regime.

One indication of these correlations is the distribution of "clump" sizes. By a clump we mean a group of particles moving in the same direction which is bracketed on the left and right by oppositely moving particles. The size of a clump is the number of particles in it. It is easy to see that if there are P particles per length and the velocities are *uncorrelated* then the number of clumps per length of size s is $n(s) = (P/2)(1/2)^2$. For instance, what is the probability that a particle is a right-moving clump of size $s=1$? With probability $\frac{1}{2}$ the particle itself is moving to the right. If velocities are uncorrelated then the probability that both its neighbors are moving to the left is $\frac{1}{4}$. We conclude that $\frac{1}{8}$ of the particles are in right-going clumps of size 1 and by symmetry $\frac{1}{4}$ of the particles are in clumps of size 1. This reasoning is easily extended to obtain the clump-size density mentioned above.

Analysis of several simulations shows that $n(s)=(P/6)(2/3)^s$ is an excellent fit to the clump-size distribution. Thus, if a particular particle is moving to the left there is a better than even chance that its neighbors are headed in the same direction, i.e., velocities are correlated. The explanation must be that collisions between oppositely directed particles occur more frequently than between particles going in the same direction. Unfortunately this trivial observation does not easily translate into an explanation of the strikingly simple distribution of clump sizes seen in the simulations.

We emphasize that these correlations are likely to be pathologies of one dimension and it is possible that a generalized Boltzmann equation is valid if $D \gtrsim 2$. This requires that the volume fraction occupied by the spheres, which is a conserved nondimensional quantity, is small. (Of course, in one dimension this fraction is not dynamically important.)

As a simple model of the development of fluctuations away from a boundary we introduce a spatially inhomogeneous version of our original ballistic agglomeration problem. We imagine a "machine gun" that fires bullets along the x axis. The bullets leave the muzzle at $x=0$ with mass m_0 at intervals τ. However, the velocities are random with some pair-distribution function, $\mathcal{P}(u)$, so that there is a well-defined mean velocity $\bar{u} \equiv \int_0^\infty u\mathcal{P}(u)\,du$. The faster bullets overtake and collide inelastically with their neighbors. Mass and momentum are conserved by these binary collisions, and the participants stick together forming clusters. The result is aggregates whose mass increases with distance from the muzzle. How does the expected mass scale with this distance x?

The problem is interesting because there is a plausible, but incorrect, scaling argument with many fluid mechanical analogs. Numerical simulation shows that the hydrodynamic scaling is wrong and the alternative is right.

The hydrodynamic scaling argument begins correctly by observing that both the flux of mass and the flux of momentum are independent of x and so can be found from conditions at the muzzle. Thus the flux of mass is $F_m=m_0/\tau$ and the flux of momentum is $F_p=m_0\bar{u}/\tau$. The incorrect assumption is that at large distances from the muzzle, where there have been many collisions, the only properties of the initial conditions that can be important are these two conserved fluxes and the distance x. One is asserting that at large x there is a universal distribution of masses and velocities whose structure is independent of all details of $\mathcal{P}(u)$ except $\bar{u}=F_p/F_m$.[9] From the three quantities F_m, F_p, and x there is only one combination with the dimensions of mass: $m(x) \sim F_p^2 x/F_m$. This scaling argument is specious and one clue is that it predicts that the velocity fluctuations are

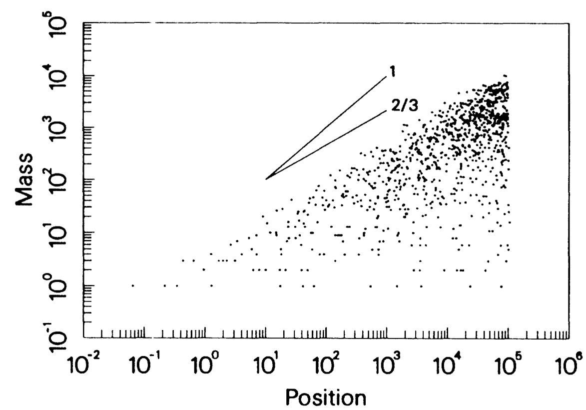

FIG. 2. Mass against position in a "machine gun" simulation in which 10^6 shots were fired at intervals of $\tau=0.1$. The muzzle velocities are uniformly distributed in the interval $(\frac{1}{2},\frac{3}{2})$. At this time, $t=100\,000$, there are 915 particles.

independent of x. This is because F_p/F_m has dimensions of velocity so that we are forced to the unlikely conclusion that after a large number of collisions the velocity fluctuations become independent of x and of order F_p/F_m.

A different argument simply replaces time in the earlier relations by $x/\bar{u}$ and a_0 by $\bar{u}\tau$. This gives

$$m(x) \sim m_0 \left(\frac{x u_{\mathrm{rms}}(0)}{\tau \bar{u}^2} \right)^{2/3},$$
$$u_{\mathrm{rms}}(x) \sim u_{\mathrm{rms}}(0) \left(\frac{\tau \bar{u}^2}{x u_{\mathrm{rms}}(0)} \right)^{1/3}. \tag{9}$$

While the hydrodynamic scaling is certainly wrong, it is not entirely clear that Eq. (9) is right. Figure 2 shows the results of a simulation in which 1 000 000 shots were fired at intervals of $\tau=0.1$. At $t=100\,000$ there are 915 aggregates and the $x^{2/3}$ increase of the expected mass is clear. Remarkably, even at very large distances from the muzzle there are still particles that have not had a single collision.

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[9]One objection is that if there are no velocity fluctuations at the muzzle (i.e., all the velocities are equal to $\bar{u}$) then there are no collisions and there is no universal scaling. In reply to this, one could argue that the development of the universal regime requires a certain number of collisions. As the initial velocity fluctuations are reduced, the distance bullets travel before these collisions take place increases so that in the limit an infinite distance is required for the universal regime to be realized.