Becchi-Rouet-Stora-Tyutin Cohomology of Compact Gauge Algebras

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We discuss the Becchi-Rouet-Stora-Tyutin (BRST) cohomology of compact Lie algebras and their infinite extensions in gauge theories. The co-BRST operator is used to construct a BRST-invariant operator W, the zero modes of which are in one-to-one correspondence with the BRST cohomology. Nontrivial solutions to the BRST cohomology conditions differing only in ghost content are shown to exist. A possible connection with supersymmetric topological quantum theories is observed.

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The essential property of supersymmetry, in both relativistic[1,2] and nonrelativistic[3,4] applications, is that its generator can be considered as the square root of the Hamiltonian:

$$\{Q,Q^\dagger\}=H\,, \tag{1}$$

where Q is the supercharge and H is the Hamiltonian of the supersymmetric system. The braces denote graded Poisson brackets in classical mechanics, or an anticommutator in quantum mechanics.

It is known that the Becchi-Rouet-Stora-Tyutin (BRST) operator[5,6] Ω may in a similar way be viewed as the square root of an invariant operator W (the BRST-extended quadratic Casimir operator) of a Lie algebra.[7,8] The BRST cohomology is then defined by the zero modes of this operator W.

Consider a compact Lie algebra $\mathcal{G}$ with n generators G_α, $\alpha=1,\ldots,n$:

$$[G_\alpha,G_\beta]=if_{\alpha\beta}{}^{\gamma}G_\gamma\,.$$

Introducing ghost variables (c^α,b_α) with canonical brackets

$$\{c^\alpha,b_\beta\}=\delta^\alpha_\beta\,,$$

we can define the BRST operator (for reviews, see, for example, Refs. 9–13)

$$\Omega=c^\alpha G_\alpha+\frac{i}{2}c^\gamma c^\beta f_{\beta\gamma}{}^{\alpha}b_\alpha\,.$$

This operator is nilpotent:

$$\Omega^2=0\,.$$

It acts in the state space of the ghosts, which is represented by polynomials in the ghost variables c^α:

$$\psi=\sum_{k=0}^{n}\frac{1}{k!}c^{\alpha_1}\cdots c^{\alpha_k}\psi^{(k)}_{\alpha_1\cdots\alpha_k}\,, \tag{2}$$

the states $\psi^{(k)}$ of ghost number k taking values in some representation space of $\mathcal{G}$. The ghosts (c^α,b_α) and the BRST operator are self-adjoint with respect to the indefinite inner product[14]

$$\langle\phi,\psi\rangle=\int[dc^n\cdots dc^1]\phi^\dagger\psi\,.$$

In addition, there also exists a positive-definite scalar product

$$(\phi,\psi)=\langle\mathcal{P}\,{}^*\phi,\psi\rangle$$
$$=\sum_{k=0}^{n}\frac{1}{k!}\phi^{\dagger(k)\alpha_1\cdots\alpha_k}\psi^{(k)}_{\alpha_1\cdots\alpha_k}\,,$$

where ${}^*\phi$ is the dual of ϕ in the sense of the Hodge star operation, and $\mathcal{P}$ denotes the reversal of the order of the ghosts in ψ, Eq. (2). With respect to this scalar product the ghost c^α and its momentum b_α are not self-adjoint, but conjugate to one another. Then the BRST operator is not self-adjoint with respect to the scalar product, but we obtain instead

$$\Omega^\dagger=G^\alpha b_\alpha+\frac{i}{2}c^\alpha f_\alpha{}^{\beta\gamma}b_\gamma b_\beta,\quad \Omega^{\dagger 2}=0\,.$$

$\Omega^\dagger$ is referred to as the co-BRST operator. The anticommutator of Ω and $\Omega^\dagger$ defines a BRST- and co-BRST-invariant extension of the quadratic Casimir operator of the Lie algebra $\mathcal{G}$:[7,8]

$$W=\{\Omega,\Omega^\dagger\}=G_\alpha^2+\text{ghost terms}\,. \tag{3}$$

Indeed, the nilpotence of Ω and $\Omega^\dagger$ together with the Jacobi identities imply

$$[\Omega,W]=0,\quad[\Omega^\dagger,W]=0\,. \tag{4}$$

The analogy between Eq. (3) and the supersymmetry algebra (1) is obvious.

The cohomology of Ω is defined as the set of equivalence classes of states which are BRST invariant while differing only by a BRST transformation:

$$\Omega\psi=0\,,$$

with

$$\psi\sim\psi'\Longleftrightarrow\psi'=\psi+\Omega\chi\,,$$

for arbitrary χ. This space is denoted by $H(\Omega)$:

$$H(\Omega)=\mathrm{Ker}\,\Omega/\mathrm{Im}\,\Omega\,.$$

The space $H(\Omega)$ is usually interpreted to define the physical states of a system subject to first-class constraints with the algebra $\mathcal{G}$.[9-13,15] In the present case this system is a quantum gauge theory restricted to spatially constant configurations.

Following a standard terminology, BRST-invariant states ψ are called *BRST closed*, and BRST-transformed states (such as $\Omega\chi$) are *BRST exact*. Every BRST-exact state is closed, but the inverse is not necessarily true. Nonzero elements of $H(\Omega)$ correspond to nonexact closed states. By definition, *BRST-harmonic* states are zero modes of W,

$$W\psi=0\,, \tag{5}$$

which holds if and only if ψ is BRST and co-BRST invariant:

$$\Omega\psi=0,\quad \Omega^{\dagger}\psi=0\,. \tag{6}$$

To prove this, it suffices to observe that

$$(\psi,W\psi)=(\Omega\psi,\Omega\psi)+(\Omega^{\dagger}\psi,\Omega^{\dagger}\psi)\,,$$

showing that W is a positive-definite operator and that it has a zero mode only if conditions (6) are satisfied.

From the properties of $\Omega,\Omega^{\dagger}$ and W it follows that *any state* $\psi[c]$ *can be decomposed into a BRST-exact, a BRST-coexact, and a BRST-harmonic state*:

$$\psi=\omega+\Omega\chi+\Omega^{\dagger}\phi\,, \tag{7}$$

where ω is BRST harmonic. This property is known as the Hodge decomposition theorem.[16]

The decomposition theorem implies that there is a one-to-one correspondence between the cohomology $H(\Omega)$ and the harmonic states. To see this, note that Eq. (7) implies

$$\Omega\psi=\Omega\Omega^{\dagger}\phi\,.$$

But then the condition of BRST invariance, $\Omega\psi=0$, leads to

$$(\Omega^{\dagger}\phi,\Omega^{\dagger}\phi)=(\phi,\Omega\psi)=0\,.$$

Hence the BRST-coexact state $\Omega^{\dagger}\phi$ must vanish, and

$$\psi=\omega+\Omega\chi\,.$$

It follows that any BRST-closed state is equivalent to a BRST-harmonic state modulo a BRST-exact state. In the language of field theory this is expressed by the statement that one can always choose a gauge in which a BRST-invariant state is also co-BRST invariant.

Using the decomposition theorem and the equivalence between the BRST cohomology and the zero modes of W, we can prove the following result.

All BRST cohomology classes are represented by states $\psi[c]$ *which satisfy*

$$G_a\psi[c]=0,\ \text{and}\ \Sigma_a\psi[c]=0\,. \tag{8}$$

Here

$$\Sigma_a=-if_{a\beta}{}^{\gamma}c^{\beta}b_{\gamma}$$

defines a representation of $\mathcal{G}$ in the space of ghosts. The proof of the conditions (8) is obtained by writing out W:

$$W=(\Omega+\Omega^{\dagger})^2=\tfrac{1}{2}G_a^2+\tfrac{1}{2}(G_a+\Sigma_a)^2\,.$$

Being a sum of squares of Hermitian operators, W can vanish only if the terms vanish separately, as implied by Eqs. (8).

The first condition (8) states that the states $\psi^{(k)}$ must be $\mathcal{G}$ invariant. This reproduces the first-class constraints on physical states. The second condition reads, in components,

$$f^{\gamma}_{a[\alpha_1}\psi^{(k)}_{\alpha_2\cdots\alpha_k]\gamma}=0\,,$$

where the square brackets denote complete antisymmetrization over the enclosed indices. This equation states that the zero modes $\psi^{(k)}$ of ghost number k are given by all invariant antisymmetric tensors of rank k, including the trivial ones with $k=0$ or n. Note that for semisimple Lie algebras a nontrivial solution always exists for $k=3$, in the form of states proportional to the structure constants:

$$\psi^{(3)}_{\alpha\beta\gamma}=f_{\alpha\beta\gamma}\chi\,, \tag{9}$$

where χ is any G singlet. This follows from the Jacobi identity. Other solutions may exist, depending on the algebra. Thus we have established that the BRST cohomology of semisimple Lie algebras is nontrivial and that several copies of the physical Hilbert space of states may exist in BRST-quantized gauge theories, differing only in the ghost number of the states.

Extension of these results to infinite Lie algebras connected with gauge theories in $(d+1)$-dimensional space-time is possible. We only present some of the more relevant equations here. Introducing ghost operators $c^{a}(x,t),b_a(x,t)$ with an equal-time anticommutator

$$\{c^{a}(x,t),b_{\beta}(y,t)\}=\delta^{a}_{\beta}\delta^{d}(x-y)\,, \tag{10}$$

we can construct the generators

$$\Sigma_a(x,t)=-if_{a\beta}{}^{\gamma}c^{\beta}(x,t)b_{\gamma}(x,t)\,,$$

which define a representation of the infinite-dimensional Lie algebra

$$[\Sigma_a(x,t),\Sigma_{\beta}(y,t)]=if_{a\beta}{}^{\gamma}\Sigma_{\gamma}(x,t)\delta^{d}(x-y)\,. \tag{11}$$

This representation is anomaly-free with respect to a ghost vacuum state $|0\rangle_{\mathrm{gh}}$ such that

$$b_a(x,t)\,|0\rangle_{\mathrm{gh}}=0\,.$$

This choice is consistent with Eq. (10) and allowed because the b ghosts anticommute with themselves. It is a natural choice in the Weyl gauge $A_0=0$, in which positivity of the Hamiltonian in the extended state space is thereby respected. The BRST cohomology contains gauge-singlet states subject to the condition

$$\Sigma_a(x,t)\psi[c]=0\,,$$

which has in addition to the trivial solution $\psi[c]\sim|0\rangle_{\rm gh}$ with ghost number zero also nontrivial solutions analogous to (9):

$$\psi[c]=\frac{1}{3!}f_{\alpha\beta\gamma}\int d^dx\,[c^\alpha(x,t)c^\beta(x,t)c^\gamma(x,t)]\,|0\rangle_{\rm gh}\otimes|\chi\rangle\,.$$

Here $|\chi\rangle$ is any gauge-singlet state with respect to the other degrees of freedom.

A different but potentially interesting situation occurs in $d=1$ if we have chiral ghost vacua, as in the light-cone gauge $A_+=0$ when

$$b_+(z)|0,+\rangle=0,\quad c_+(z)|0,+\rangle=0\,.$$

Here $c_+(z)$ denotes the positive-frequency part of $c(z)$, while $b_+(z)$ denotes the non-negative frequency modes of $b(z)$. Then the loop algebra (11) picks up a Schwinger term and becomes a Kac-Moody algebra. In the BRST construction the central charge of the ghosts is now canceled by a contribution from the unphysical gauge sector, which has the opposite sign. (The necessity of this result was noted in Refs. 17 and 18.) After normal ordering with respect to the field operators and the Kac-Moody generators, the generalized quadratic Casimir operator becomes

$$W=\tfrac{1}{2}\int dx\,\{:G_a(x)^2:+:[G_a(x)+\Sigma_a(x)]^2:-i:c^a(x)\overleftrightarrow{\partial}_x b_a(x):\}\,.$$

Observe that the integrand of W is a linear combination of two Virasoro currents constructed according to the Sugawara prescription and a Virasoro current corresponding to a ghost energy-momentum tensor.

Finally, we return once more to the similarity between the BRST algebra (3) and supersymmetry, Eq. (1). The analogy is complete for theories in which the Hamiltonian is identical to W. This is a consistent choice, since W is positive definite. For such theories the Hamiltonian is BRST exact and coexact:

$$H=W=\{\Omega,\Omega^\dagger\}\,. \tag{12}$$

Hence the Hamiltonian is BRST equivalent to zero, and no dynamics results for the physical states. This is precisely the situation one encounters in topological field theories.[19] For the class of topological quantum theories (12) the full identification of BRST invariance with supersymmetry seems possible. Remarkably, reinterpreting the operator Ω as the supercharge, the algebra (12) is precisely the algebra of the zero modes of the constraints of a supergravity theory. Thus we expect a relation between supergravity and theories with local BRST invariance.[20]

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[1] Yu. A. Gol'fand and E. P. Likhtman, Zh. Eksp. Teor. Fiz. **13**, 452 (1971) [JETP Lett. **13**, 323 (1971)].

[2] J. Wess and B. Zumino, Nucl. Phys. **B70**, 39 (1974).

[3] E. Witten, Nucl. Phys. **B188**, 513 (1981).

[4] P. Salomonson and J. W. van Holten, Nucl. Phys. **B196**, 509 (1982).

[5] C. Becchi, A. Rouet, and R. Stora, Ann. Phys. (N.Y.) **98**, 28 (1976).

[6] I. V. Tyutin, Lebedev Institute Report No. FIAN 39, 1975 (unpublished).

[7] J. Gervais, Nucl. Phys. **B276**, 339 (1986).

[8] I. Bars and S. Yankielowicz, Phys. Rev. D **35**, 3878 (1987).

[9] K. Fujikawa, Prog. Theor. Phys. **63**, 1364 (1980).

[10] T. Kugo and I. Ojima, Prog. Theor. Phys. Suppl. **66**, 1 (1979).

[11] M. Henneaux, Phys. Rep. **126C**, 1 (1985).

[12] L. Beaulieu, Phys. Rep. **129C**, 1 (1985).

[13] J. W. van Holten, in *Functional Integration, Geometry and Strings*, Proceedings of the Twenty-Fifth Karpacz Winter School of Theoretical Physics, edited by Z. Haba and J. Sobczyk (Birkhäuser, Boston, 1989).

[14] F. A. Berezin, *The Method of Second Quantization* (Academic, New York, 1966).

[15] T. Kugo and I. Ojima, Phys. Lett. **73B**, 459 (1978).

[16] A proof in the context of BRST cohomology is described in J. W. van Holten, NIKHEF-H Report No. NIKHEF-H/90-6 [Nucl. Phys. B (to be published)].

[17] Z. Hlousek and K. Yamagishi, Phys. Lett. B **173**, 65 (1986).

[18] D. Chang and A. Kumar, Phys. Rev. D **35**, 1388 (1987).

[19] E. Witten, Commun. Math. Phys. **117**, 353 (1988).

[20] J. W. van Holten, Nucl. Phys. **B315**, 740 (1989).