Inhomogeneity and the Onset of Inflation

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We study numerically the onset of inflation under inhomogeneous initial conditions for various inflationary scenarios and various initial parameters. We solve the full Einstein equations for the gravitational field coupled to scalar-field and radiation-field sources, under the assumptions of spherical symmetry and a closed-universe background. We find that a large initial inhomogeneity suppresses the onset of inflation, and that the scalar field must have values suitable for inflation over a region of several horizon sizes for the inflationary phase to begin.

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The inflationary scenario holds in it the potential of freeing the present state of the Universe from extreme dependence on the initial data.[1] However, because of technical reasons, most investigations of inflationary cosmological models have assumed Robertson-Walker symmetry (i.e., homogeneity and isotropy), or small perturbations around it. In this case, one finds that inflation causes a large increase in the horizon size of the present Universe on account of the exponential rate of expansion. Several authors[2] have studied particular cases of anisotropic models and found that the scenario predicted by the Robertson-Walker model is essentially unchanged even when large anisotropy was present before the inflationary period. However, it is not obvious that cosmological models with inhomogeneous initial conditions will even enter (or "gracefully exit") an inflationary epoch, nor is it obvious that the inflation will smooth out large initial inhomogeneities.

In order for inflation to occur, the energy-momentum of the Universe must be dominated by the potential $V(\Phi)$ of the scalar field Φ that drives the inflation. It is possible to estimate qualitatively how large the initial inhomogeneity can be so that the potential term will still dominate. We define Δ to be a "comoving-coordinate" measure of the inhomogeneity (i.e., Δ is the "comoving wavelength" of the initial perturbation) and $\delta\Phi$ to be a typical change in Φ. During inflation, $H^2 \approx 8\pi V/3m_{\rm Pl}^2$ and the requirement $(\delta\Phi/R\Delta)^2 < V$ yields

$$R\Delta > \left(\frac{8\pi}{3m_{\rm Pl}^2}\right)^{1/2} H^{-1}\delta\Phi\,, \tag{1}$$

where R is the scale factor, $H=\dot{R}/R$, and H^{-1} is the horizon size [the factors 8π and $m_{\rm Pl}^2$ were included in Eq. (1) for clarity; from now on we use units in which $m_{\rm Pl}^2 = 8\pi$]. To obtain a better quantitative estimate of the influence of large inhomogeneity on inflation, one must solve the complete set of Einstein equations coupled to a massive scalar field.

With this objective in mind we have constructed a general-relativistic spherically symmetry code for solving Einstein equations coupled to a massive scalar field.[3] Unlike perturbation calculations[4] we solve the full nonlinear set of equations for both the source and the gravitational field. We decided to solve 1D spherically symmetric cosmology, since such perturbations resemble (topologically) realistic perturbations more than any other 1D model.[5]

We work in the context of a closed universe (which for simplicity has a reflection symmetry around $\pi/2$) because only in this case is it obvious what the boundary conditions are. Inflationary scenarios[6] assume the existence of large enough regions in which conditions are appropriate for entering an inflationary phase. These regions undergo an exponential expansion, and their evolution does not depend on the global topology.

We write the metric in the form

$$ds^2 = -(N^2 - R^2\beta^2)dt^2 + 2R\beta\, d\chi\, dt + R^2(d\chi^2 + \sin^2\chi\, d\Omega^2)\,, \tag{2}$$

where $0 \le \chi \le \pi$, and R, N, and β are functions of χ and t. In this calculation we use the gauge $N=1$.

The source terms are a "massive" scalar field Φ (by "massive" we mean here a scalar field with nonvanishing potential) and a radiation field Ψ which we describe by a massless scalar field. This representation enables us to couple the fields both gravitationally and thermally easily and self-consistently. We study both "new inflation"[5] and "chaotic inflation."

Our initial data are a ${\rm Tr}K=$const slice (K_{ij} is the extrinsic curvature tensor and ${\rm Tr}K$ corresponds roughly to the expansion rate[7] H) with an isocurvature initial density. The initial $\rho_{\rm total}$ (the total energy density) is constant, but the fields are inhomogeneous; i.e., the total energy at different places has different composition of a scalar field and radiation field. In all the cases, the radiation field is damped quickly, and we are left with an inhomogeneous massive scalar field. The field Φ has an initial Gaussian distribution:

$$\Phi_{\rm init}(\chi) = \Phi_0 + \delta\Phi\left[1 - \exp\left(-\frac{\sin\chi^2}{\Delta^2}\right)\right], \tag{3}$$

TABLE I. Initial parameters and results for the five cases described in Fig. 1. $R_{\rm end}^{\rm hom}$ is the scale factor at the end of inflation (when $\Phi=\Phi_{\rm end}$) of an equivalent homogeneous universe that begins with $\rho_{\rm total}$, $\Phi=\Phi_0$, $R_{\rm init}$, and $H_{\rm init}^{-1}$. $(R\Delta/H^{-1})_{\rm est}$ is an estimate [based on Eq. (1)] for the value of $R\Delta/H^{-1}$ below which inflation does not take place.

Case	*a*	*b*	*c*	*d*	*e*	*f*
$V(\Phi)$	Massive $m=0.1$	Φ^4 $\lambda=10^{-3}$	Φ^4 $\lambda=10^{-3}$	Φ^4 $\lambda=10^{-2}$ $\sigma=4$	Φ^4 $\lambda=10^{-2}$ $\sigma=4$	CW $\lambda=10^{-1}$ $\sigma=0.2$
$\rho_{\rm total}$	2.3	2.375	3.56	1.32	1.32	2×10^{-5}
$R_{\rm init}$	6	6	6	10	10	2000
$H_{\rm init}^{-1}$	1.16	1.144	0.928	2.18	2.18	395
Φ_0	6	9.5	10.5	0.001	0.001	5×10^{-4}
$\delta\Phi$	-2	-2	-3	0.5	0.1	0.01
$\Phi_{\rm end}$	~0.3	~2	~4	~0.355	~0.355	0.2
$R_{\rm end}^{\rm hom}$	2.5×10^5	1.3×10^6	6.5×10^6	4.7×10^{13}	4.7×10^{13}	7×10^{20}
$(R\Delta/H^{-1})_{\rm est}$	1.15	1.15	1.73	0.29	0.058	0.0058

where $0\le\chi\le\pi$ and the usual radial coordinate $r=\sin\chi$. We set $\dot\Phi_{\rm init}=0$. The initial distribution for the radiation field $\Psi_{\rm init}$ is given by $\rho_{\rm total}-\rho(\Phi_{\rm init})$. The distribution of the Φ field depends on three parameters: Φ_0, the value of the field at the origin $[\Phi_{\rm init}(\chi=0)=\Phi_0]$, $\delta\Phi$, which determines the value of Φ at the other end of the Universe (i.e., $\Phi_{\rm init}(\chi=\pi/2)\equiv\Phi_{\pi/2}=\Phi_0+\delta\Phi[1-\exp(-1/\Delta^2)]$), and Δ, which determines the size of the initial inhomogeneity.

One can divide the initial conditions into two types. In the first type, Φ_0 and $\delta\Phi$ are such that the values $V(\Phi_0)$ and $V(\Phi_{\pi/2})$ are both "suitable for inflation" (by suitable we mean that a homogeneous universe with such initial conditions will inflate).[8] We find that universes of this type will enter an inflationary phase everywhere. The inhomogeneity, in these cases, significantly reduces the duration of inflation but it does not suppress it completely.

In the rest of the paper we focus on initial data of the second type. In this type, the values of Φ_0 and $\delta\Phi$ are such that $V(\Phi_0)$ will have a value suitable for inflation while $V(\Phi_{\pi/2})$ will not.[9] In this case the question whether the region around the origin will inflate depends on Δ. Keeping the initial $\rho_{\rm total}$ unchanged, we vary the size of the initial inhomogeneity (i.e., Δ) and we calculate how this influences the inflationary epoch.

We consider "chaotic" inflation with $V(\Phi)=m^2\Phi^2/2$ (case *a*) and with $V(\Phi)=\lambda\Phi^4/4$ (cases *b* and *c*) and "new inflation" with a quartic potential[10] $V(\Phi)=\lambda\times(\Phi^2-\sigma^2)^2/4$ (cases *d* and *e*) and with a Coleman-Weinberg (CW) type of potential $V(\Phi)=\lambda\Phi^4[\ln(\Phi^2/\sigma^2)-\frac{1}{2}]+\lambda\sigma^4$ (case *f*). We choose parameters for the initial fields (see Table I) and we follow the evolution of the Universe until it exits the inflationary phase everywhere [when $\Phi(\chi=0)=\Phi_{\rm end}$]. For "chaotic inflation" we simply require that $V(\Phi_{\rm end})$ will be sufficiently small. For new inflation we stop when Φ starts to roll off rapidly. Figure 1 displays a graph of the expansion factor at the origin as a function of the initial $R\Delta/H^{-1}$ for the various cases, and Figs. 2 and 3 describe, for cases *a* and *d*, the trajectory of the solution at the origin in the $\bar\Pi$, TrK plane [$\Pi\equiv-(R^3/N)(\dot\Phi-\beta\Phi')$, see Eq. (2), is the conjugate momentum to Φ, and $\bar\Pi\equiv\Pi/R^3$]. The overall behavior of the corresponding curves for cases *b* and *c* resembles the curves of Fig. 2 and those of case *e* resemble the curves of Fig. 3. Figures 2 and 3 give a qualitative picture of the deviation of the solution from the corresponding homogeneous solution and of the duration of the inflationary period at the origin. Inflation is manifested by a $\bar\Pi\simeq$const phase. It is obvious that in all the cases the calculation stops long after the inflationary phase is over, and that the Universe does not inflate when Δ is small.

We can see in Fig. 1 that small inhomogeneity tends to reduce the amount of inflation and large enough inhomogeneity, $R\Delta<2H^{-1}$, does not allow the development of any inflationary period at all. Table I gives an estimate [using Eq. (1)] for the value of $R\Delta$ above which inflation takes place. Comparisons with Fig. 1 show

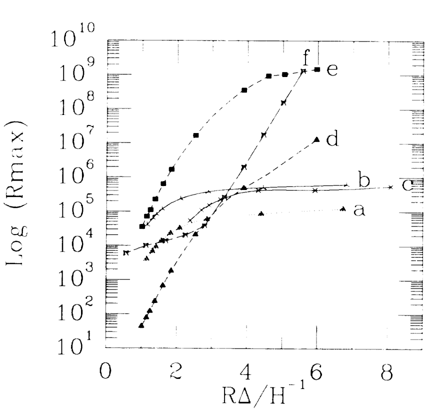

FIG. 1. The scale factor at the origin at the end of the computation as a function of the proper width of the initial Gaussian relative to the horizon size. The various curves correspond to the columns of Table I.

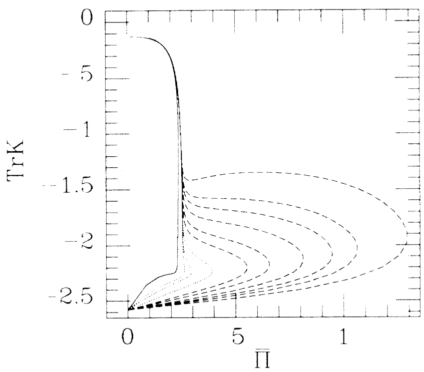

FIG. 2. TrK vs $\bar{\Pi}$ trajectories of a "chaotic inflation" (case a) solution for a homogeneous solution (solid line), small-inhomogeneity (large-Δ) solutions that undergo inflation (dotted lines), and large-inhomogeneity (small-Δ) solutions that do not undergo inflation (dashed lines). Note that the dashed lines reach the $\bar{\Pi}\approx 0.25$ phase with small TrK, i.e., a small expansion rate, and hence do not inflate. The curves correspond to the points marked on curve a of Fig. 1.

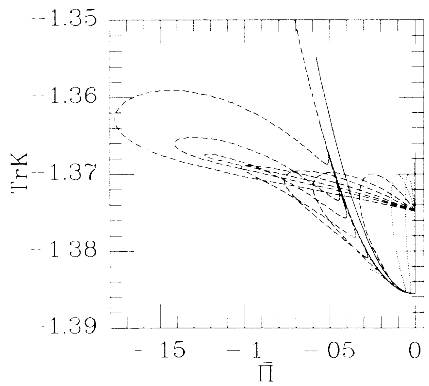

FIG. 3. Same as Fig. 2, for case d. Note that the inflationary phase is manifested at the right bottom corner of the homogeneous curve.

that, as expected, this estimate is quite good for chaotic inflation. However, for new inflation Eq. (1) underestimates the critical value by a large factor. For comparison, we have calculated $R_{\rm end}^{\rm hom}$, which is the scale factor at the end of inflation (when $\Phi=\Phi_{\rm end}$), of a homogeneous universe with the same initial conditions as the ones that are given at the origin of our inhomogeneous solutions. For new inflation, $R_{\rm end}^{\rm hom}$ is much larger than the inhomogeneous $R_{\rm end}$. In this case the Universe spends most of the inflationary period with $\Phi\approx 0$. The deviations from homogeneity induce variation of $\Phi_{\rm init}$ which speed up the roll down of Φ from the origin.

We see the same behavior when we repeat the calculations with smaller coupling constants (m or λ). Details of these and other calculations will be published elsewhere.

In an inhomogeneous universe,

$$H=\left[H_{\rm hom}^2+\frac{\Phi'^2}{6R^2}+\frac{2}{R^3}\left(R''+2R'\cot\chi-\frac{R'^2}{R}\right)\right]^{1/2}, \tag{4}$$

where a prime denotes a derivative with respect to χ. $H\approx \dot{R}/R$ acts as a friction term in the evolution equation for the massive scalar field. An increase in $\dot{R}/R$ slows down the evolution of the Φ field, causing it to spend more time on the flat part of the potential, and this increases the duration of inflationary phase. This has led to the suggestion[11] that inhomogeneity will increase the duration of inflation. However, only a full solution of the whole set of inhomogeneous relativistic equations can determine how the introduction of inhomogeneity will influence the term $\dot{R}/R$. Our numerical results show that the overall effect of small inhomogeneities is a decrease in the amount of inflation. When the inhomogeneity is large there is a qualitative change in the overall behavior of the solution (which is apparent in the trajectories of Figs. 2 and 3) and the Universe does not enter an inflationary period at all.

We have not attempted an exhaustive classification of the relation between initial inhomogeneities and inflation for all possible inflationary potentials and all possible parameters. Furthermore, because of practical limitations, we have not used realistic parameters to achieve the amount of inflation needed in realistic scenarios. We expect that neither of the above objections, subject to the previous remarks, will change our basic conclusions. In all the cases that we have examined we found that the scalar field must have values suitable for inflation over a region of the size of several horizons in order to get inflation. In other words, it seems that modes with wavelength larger than three horizons must be excited in order to get inflation. Such an initial homogeneity might or might not be "natural" in different cosmological scenarios.[12] Once inflation begins, it can increase the size of the horizon by huge factors and solve the current "large-number" horizon problem.

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[1]For a recent review of inflation, see M. S. Turner, in

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[5]H. Kurki-Suonio, J. Centrella, R. A. Matzner, and J. R. Wilson, Phys. Rev. D **35**, 435 (1987), studied numerically "new inflation" under planar symmetry. We are considering here both "chaotic" and "new" inflation under spherical rather than planar symmetry. Spherical configurations behave gravitationally quite differently from infinite planar ones and are more natural.

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[7]T. Piran, in *The Early Universe,* edited by W. Unruh (Reidel, Dordrecht, 1988).

[8]In this case $\delta\Phi$ must be smaller than a typical value of the scalar field Φ_0 for chaotic inflation or σ for new inflation.

[9]In this type of initial data for "chaotic" inflation with small coupling constants the initial gradients are essentially larger than the potential term. When the potential (with a small coupling constant) term is comparable to the gradient terms ($\delta\Phi$ is essentially $\ll\Phi$) the initial data are of the previous type; i.e., the conditions everywhere are suitable for inflation.

[10]We find that inhomogeneity has roughly the same effect on inflation with a quartic potential and on inflation with a CW-type potential, provided that they are of similar "flatness." Reference 5 finds that inhomogeneity influences inflation with a quartic potential much more than inflation with a CW-type potential, but they compare potentials with similar parameters, and in this case the quartic potential is much steeper than the CW one.

[11]L. G. Jensen and J. A. Stein-Schabes, Phys. Rev. D **35**, 1146 (1987); J. A. Stein-Schabes, Phys. Rev. D **35**, 2345 (1987).

[12]In the new inflation scenario the value of Φ could be driven towards $\Phi\approx 0$ at spacelike-separated regions in the pre-inflationary era.