Colored Black Holes

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We analyze the static spherically symmetric Einstein-Yang-Mills equations with SU(2) gauge group and show numerically that the equations possess asymptotically flat solutions with regular event horizon and nontrivial Yang-Mills (YM) connection. The solutions have zero global YM charges and asymptotically approximate the Schwarzschild solution with quantized values of the Arnowitt-Deser-Misner mass. Our result questions the validity of the "no-hair" conjecture for YM black holes. This work complements the recent study of Bartnik and McKinnon on static spherically symmetric Einstein-Yang-Mills soliton solutions.

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Recently, Bartnik and McKinnon have found numerically a family of soliton solutions of the Einstein-Yang-Mills (EYM) equations with the SU(2) gauge group.[1] This was rather unexpected since many facts suggested that such solutions did not exist. In particular, neither the vacuum Einstein equations nor the Yang-Mills equations have nontrivial static globally regular solutions.[2]

In this Letter we show numerically that, similarly to the globally regular case, the static SU(2) EYM equations also possess spherically symmetric black-hole solutions with nontrivial Yang-Mills (YM) connection. The existence of such black-hole solutions is quite surprising. In the previous studies of the EYM equations in the black-hole sector, the only solutions which were found had zero YM curvature. These effectively Abelian solutions are given by the Kerr-Newman metric with the trivial Coulomb-type YM connection.[3] It was even conjectured that these U(1) solutions are the only solutions of the EYM equations; i.e., black holes have no non-Abelian YM hair. Our result shows that this conjecture was false and the EYM equations have a much richer structure than the Einstein-Maxwell equations.

Let us point out that our result questions the validity of the "no-hair" conjecture for YM black holes. This conjecture, well established for linear matter fields by uniqueness theorems for the Kerr-Newman black holes[4] and nonexistence results of Bekenstein,[5] states that the structure of a stationary black hole is completely determined by global charges defined at spatial infinity such as Arnowitt-Deser-Misner (ADM) mass, angular momentum, or electric charge. In the case of colored black holes, the global YM charges are necessarily zero, so the ADM mass remains the only global parameter describing these solutions. The existence of such black holes is incompatible with the basic idea of the no-hair conjecture, since the YM hair is not associated with any global charge which would forbid it to be radiated away to infinity. However, let us admit that so far this does not constitute a physically serious counterexample against the no-hair conjecture because the colored black hole might be unstable and once perturbed it loses the YM hair ending up as the Schwarzschild black hole. Our work relies extensively on Bartnik and McKinnon's study and we adopt their notation from Ref. 1.

EYM equations and boundary conditions.—Let us write a static spherically symmetric metric in the form

$$ds^2=-(1-2m/r)e^{-2\delta}dt^2+(1-2m/r)^{-1}dr^2 +r^2(d\vartheta^2+\sin^2\vartheta\, d\phi^2)\,, \tag{1}$$

where the functions m and δ depend only on the Schwarzschild radius r. We want this metric to describe a nonsingular, asymptotically flat spacetime outside a regular event horizon located at $r=r_h$. This implies the following boundary conditions:

(i) Asymptotic flatness requires that as $r\to\infty$,

$$m(r)\to M=\text{const and } \delta(r)\to\delta_0=\text{const}\,. \tag{2}$$

(ii) The existence of a regular event horizon at $r=r_h$ requires that

$$2m(r_h)=r_h \text{ and } \delta(r_h)<\infty\,. \tag{3}$$

For our purposes it is convenient to put $\delta(r_h)=0$ [the usually used choice $\delta(\infty)=0$ is connected with our choice by simple time-coordinate rescaling].

(iii) Nonexistence of singularities implies that

$$2m(r)<r \text{ for } r>r_h\,. \tag{4}$$

The general spherically symmetric SU(2) connection is given by[6]

$$A=a\tau_3dt+b\tau_3dr+(w\tau_1+d\tau_2)d\vartheta +(\cot\vartheta\,\tau_3+w\tau_2-d\tau_1)\sin\vartheta\, d\phi\,, \tag{5}$$

where a, b, w, and d are functions of (r,t) and τ_i $(i=1,2,3)$ are standard generators of su(2) Lie algebra. This form of connection is preserved by U(1) gauge transformations,

$$A'=h^{-1}Ah+h^{-1}dh\,,$$

where $h=\exp[\psi(r,t)\tau_3]$. Using this gauge freedom we can set $b\equiv0$. Now, we assume further that the connection is static; i.e., a, w, and d depend only on r. In this case the Yang-Mills equations imply that $d=Cw$, where C is a constant, so that we can make a further constant gauge transformation to set $d\equiv 0$. Hence, the general static, spherically symmetric SU(2) connection is described by two functions $w(r)$ and $a(r)$. The Yang-Mills curvature $F=dA+A\wedge A$ is given by

$$\begin{aligned} F = {} & a'\tau_3 dr\wedge dt + w'\tau_1 dr\wedge d\vartheta \\ & + w'\tau_2 dr\wedge \sin\vartheta d\phi - (1-w^2)\tau_3 d\vartheta\wedge\sin\vartheta d\phi \\ & + aw\tau_2 dt\wedge d\vartheta - aw\tau_1 dt\wedge \sin\vartheta d\phi . \end{aligned} \tag{6}$$

The EYM coupled system is described by the action

$$S=\frac{1}{16\pi}\int (R-F^2)(-g)^{1/2}d^4x , \tag{7}$$

where $F^2=F_{ab}F^{ab}$. The dynamical equations derived from the action (7) are the YM equations $d^*F=0$ and the Einstein equations $G_{ab}=8\pi T_{ab}$ with the stress-energy tensor

$$T_{ab}=(1/4\pi)(F_{ac}F_{bd}g^{cd}-\tfrac{1}{4}g_{ab}F^2) . \tag{8}$$

Before we write down the equations explicitly we shall make still one more simplification. Namely, we exclude the electric part of the YM field by imposing the 't Hooft-Polyakov *Ansatz* $a\equiv 0$. In fact, Galtsov and Ershov have shown recently[7] that assuming suitable asymptotic behavior the only static black-hole solution with nonzero YM electric field is the Reissner-Nordström metric with the electric charge e and magnetic charge g:

$$\begin{aligned} & m=M-(e^2+g^2)/2r , \\ & F=(e/r^2)\tau_3 dr\wedge dt - g\tau_3 d\vartheta\wedge\sin\vartheta d\phi , \end{aligned} \tag{9}$$

where $g=0$ if $w\equiv 1$ or $g=1$ if $w\equiv 0$. Let us note, however, that Galtsov and Ershov assume too restrictive falloff conditions for the electric YM field. Bartnik[8] pointed out that, under weaker boundary conditions, there may exist nontrivial solutions with nonvanishing electric and magnetic YM fields. The existence of such dyon solutions is currently under investigation. For purely magnetic YM fields the EYM equations reduce to the system[1]

$$[e^{-\delta}(1-2m/r)w']'+e^{-\delta}(1-w^2)w/r^2=0 , \tag{10}$$

$$\delta'=-2w'^2/r , \tag{11}$$

$$m'=(1-2m/r)w'^2+\tfrac{1}{2}(1-w^2)^2/r^2 . \tag{12}$$

Now, let us consider what are the boundary conditions for the function $w(r)$. The requirement that the local energy density

$$4\pi\mu=\tfrac{1}{2}(1-w^2)^2/r^4+(1-2m/r)w'^2/r^2=m'/r^2 \tag{13}$$

(here $\mu=-T_0^0$) be bounded at $r=r_h$ puts no extra condition on w. We shall assume that w is C^1 at $r=r_h$. Asymptotic flatness requires that the total energy outside the event horizon is finite,

$$M-m(r_h)=4\pi\int_{r_h}^{\infty}\mu r^2 dr<\infty , \tag{14}$$

which means that $w\to$ const as $r\to\infty$. Assuming that w is constant at infinity we can solve Eq. (10) asymptotically to get

$$|w|\simeq 1-c/r,\quad c>0 . \tag{15}$$

For completeness let us note that Eqs. (10)-(12) also possess the trivial solution $w\equiv 0$ with $m=M-1/2r$ and $\delta=$const which corresponds to the Reissner-Nordström metric with magnetic charge $g=1$. It follows from (6) and (15) that asymptotically $F_{\vartheta\phi}\sim r^{-1}$, and so the YM magnetic charge vanishes. Consequently, a solution for which $|w|=1$ at infinity is characterized by only one global parameter: the ADM mass. Thus according to the no-hair conjecture the only solution should be the Schwarzschild solution which corresponds to $|w|\equiv 1$, $M=$const. We put much effort into trying to prove this uniqueness but failed. Then, encouraged by Bartnik and McKinnon's results, we tried numerical computations and to our surprise we found the essentially non-Abelian solutions.

Numerical solutions.— Let us make some *a priori* observations about the global behavior of solutions. From (10) it follows that if w reaches a local extremum at some radius r then at this point

$$\text{sgn}w''=\text{sgn}(w^2-1)w . \tag{16}$$

Hence w cannot have local maxima for $w>1$ and local minima for $w<-1$. From (10) we also have

$$\text{sgn}w'(r_h)=\text{sgn}(w^2-1)w|_{r_h} . \tag{17}$$

Since $|w(\infty)|=1$, it follows from (17) that $|w|\le 1$. It is easy to show that either $|w|<1$ or $|w|\equiv 1$. Now, using (11) and (12), we replace Eq. (10) by

$$r^2(1-2m/r)w''+[2m-(1-w^2)^2/r]w' + (1-w^2)w=0 . \tag{10a}$$

We numerically solve Eqs. (10a), (11), and (12) with the boundary conditions (2), (3), and (15) using the standard shooting method of solving two-point boundary-value problems.[9] For a given value r_h, we need four initial values: $\delta(r_h)$, $m(r_h)$, $w(r_h)$, and $w'(r_h)$. From (11) it is obvious that we can choose $\delta(r_h)$ arbitrarily. We take $\delta(r_h)=0$. From (3) we have $m(r_h)=\frac{1}{2}r_h$. Finally, it follows from (10a) that $w(r_h)$ and $w'(r_h)$ must satisfy the constraint

$$[2m-(1-w^2)/r]w'+(1-w^2)w|_{r_h}=0 .$$

To sum up we have only one free shooting parameter $w(r_h)$ which satisfies $|w(r_h)|<1$. For a fixed value of

r_h we choose some $w(r_h)$ and propagate it numerically trying to aim at ± 1 at infinity. For example, for $r_h=1$, if we take $w(r_h)$ a bit less than 1, say $w(r_h)=0.7$, then the solution diverges rapidly to $-\infty$. If we take $w(r_h)=0.6$, then the solution reaches a minimum for $-1<w<0$ and then diverges to $+\infty$. It turns out that there exists a value $w_1(r_h)\in(0.6,0.7)$ such that the solution goes asymptotically to -1 [Fig. 1(a)]. In a similar way we can find solutions with a greater number of zeros, which, after oscillating in a region $|w|<1$, tend asymptotically to ± 1. Thus, for a given value of r_h we get a discrete family of asymptotically flat solutions, labeled by n, the number of nodes of w. This structure is very similar to that found by Bartnik and McKinnon for globally regular solutions. However, notice that, in contrast to the regular case, we do not have a natural starting value r_h. Since EYM equations do not have the scaling symmetry (remember that we have fixed the scale by putting the YM coupling constant $\lambda=1$), *a priori* we should treat r_h as the second shooting parameter. Luckily enough, it turns out that for every value of r_h we get a similar discrete family of solutions. The initial value $w(r_h)$ changes continuously with r_h, indicating that probably there exists a hidden symmetry which allows us to transform solutions with different values of r_h one into another.

The behavior of solutions for $r_h=1$ is shown in Fig. 1. Near the event horizon the solutions are very close to the Reissner-Nordström solution. To show this we define after Ref. 1 the effective magnetic charge $g^2(r)$ by

$$e^{-2\delta}(1-2m/r)=1-2M/r+g^2/r^2.$$

Figure 1 shows that just outside the horizon $g^2(r)\simeq 1$. Further from the horizon we have a transition zone; the magnetic charge slowly decays. When w reaches its asymptotic value ± 1, the charge drops to zero and the solution approximates the Schwarzschild solution with mass M_n.

Discussion.— We realize of course that the above numerical construction gives only a strong evidence for the existence of solutions. The analytical proof of existence is obviously needed. Here, insight provided by numerical

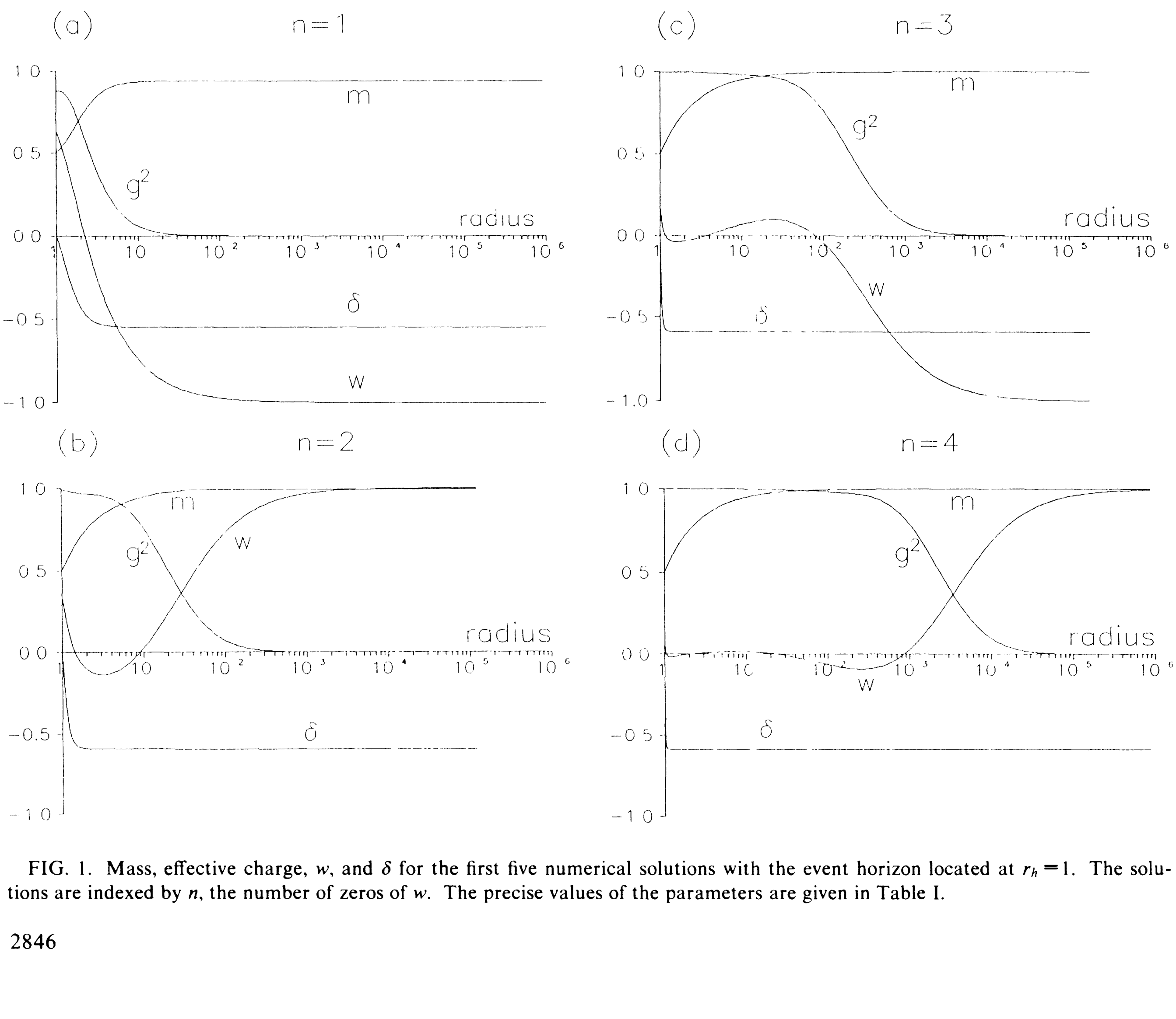

FIG. 1. Mass, effective charge, w, and δ for the first five numerical solutions with the event horizon located at $r_h=1$. The solutions are indexed by n, the number of zeros of w. The precise values of the parameters are given in Table I.

TABLE I. Shooting initial value $w_n(r_h)$, ADM mass M_n, and $\delta_n(\infty)$ for the first five numerical solutions with the event horizon located at $r_h = 1$ (n is the number of zeros of w).

n	$w_n(r_h)$	M_n	$\delta_n(\infty)$
1	0.632 206 952	0.937 191 161	−0.548 472 096
2	0.345 178 112	0.993 848 856	−0.593 180 907
3	0.187 579 799	0.999 437 943	−0.590 174 478
4	0.102 277 365	0.999 949 567	−0.587 905 574
5	0.055 839 881	0.999 995 498	−0.587 110 995

solutions should be helpful. In particular, nontrivial information about the solutions is probably hidden behind some striking numerical coincidences. For example, we do not understand why the ratio $w_n(r_h)/w_{n+1}(r_h)$ is almost constant for every n.

There are two additional questions that are relevant to whether the above solution is physical or not. The first question is concerned with the fact that the Schwarzschild coordinates we have used are pathological at the horizon. Thus it may happen that the smooth solution in r,t coordinates corresponds to one which in well behaved coordinates has singularities. To show that this does not happen for our solution we have checked the behavior of metric functions and the YM field in Kruskal-like coordinates and found that they are regular everywhere.

Another important problem is the question of stability. It is clear from the above construction that although locally our solutions depend continuously on initial data at $r = r_h$, they are not globally stable against a small change of initial data. In other words, the Cauchy problem (evolution in r with initial data at $r = r_h$) for our equations has the global solution only for a discrete set of initial data. From the mathematical point of view, this instability is nothing strange; in fact, it is quite typical for elliptic eigenvalue problems. However, physically this suggests that a balance between the attractive gravitational force and the repulsive YM force is very fragile and may be unstable in time. In particular, the solutions found by Bartnik and McKinnon are not stable under small time-dependent perturbations.[10] We hope that the presence of the horizon can stabilize the solutions. We are currently working on this problem. The preliminary analysis shows that for small time-dependent perturbations there are no exponentially growing radial modes with the correct behavior at the horizon. There is an exciting possibility that the static colored black hole, if it turns out to be stable, is the final state of the collapsed Bartnik-McKinnon soliton star.

In conclusion, let us remark that colored black holes may have interesting astrophysical applications but we would not like to speculate about it before the question of stability is settled positively.

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[1]R. Bartnik and J. McKinnon, Phys. Rev. Lett. **61**, 141 (1988).

[2]A. Lichnerowicz, in *Les Theories Relativistes de la Gravitation* (Masson, Paris, 1955); S. Deser, Phys. Lett. **64B**, 463 (1976); S. Coleman, in *New Phenomena in Subnuclear Physics*, edited by A. Zichichi (Plenum, New York, 1975).

[3]P. B. Yasskin, Phys. Rev. D **12**, 2212 (1975); M. Perry, Phys. Lett. **71B**, 234 (1977).

[4]W. Israel, Commun. Math. Phys. **8**, 245 (1968); D. C. Robinson, Phys. Rev. Lett. **34**, 905 (1977); P. Mazur, J. Phys. A **15**, 3173 (1982); B. Carter, in *Gravitation in Astrophysics*, edited by B. Carter and J. B. Hartle (Plenum, New York, 1987).

[5]J. D. Bekenstein, Phys. Rev. Lett. **28**, 452 (1972); Phys. Rev. D **5**, 1239 (1972); **5**, 2403 (1972).

[6]E. Witten, Phys. Rev. Lett. **38**, 121 (1977); J. McKinnon, Honor thesis, Australian National University, 1987.

[7]D. V. Galtsov and A. A. Ershov, Phys. Lett. A **138**, 160 (1989).

[8]Robert Bartnik (private communication).

[9]W. H. Press *et al.*, *Numerical Recipes: The Art of Scientific Computing* (Cambridge Univ. Press, Cambridge, 1985).

[10]N. Straumann and Z.-H. Zhou (to be published).