Solitary Waves Generated by Subcritical Instabilities in Dissipative Systems

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We show that stable localized waves can be generated in the vicinity of an inverted Hopf bifurcation. We compute the size of the localized wave envelope perturbatively in the case of slightly dissipative systems. The size selection traces back to the broken scale invariance by the dissipative terms. This mechanism is a possible explanation for the localized structures, widely observed in various hydrodynamic flows in dissipative systems driven far from equilibrium.

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Localized structures are widely observed in systems far from equilibrium. Well-known examples are the local regions of turbulent motion surrounded by laminar flow, which develop in many open-flow experiments (e.g., pipe flow, channel flow, boundary layers).[1] More recently, spatially localized standing surface waves have been observed on a horizontal layer of fluid submitted to vertical vibrations,[2] and convection in binary-fluid mixtures displayed localized traveling waves.[3-5] In all cases, the possible origin of localized structures lies in the existence of a subcritical instability, which implies that two different homogeneous stable states coexist in an interval range of the control parameter. The simplest spatial nonuniformity consists of an interface between the two stable states. A similar situation occurs in first-order phase transitions, for instance, when droplets of liquid nucleate in a supersaturated vapor. In phase transitions, the droplets are always unstable; they either shrink or expand. In the instability problem, a "droplet" consists of a region where the system is in the bifurcated state, surrounded by the basic state. When there exists a Lyapunov functional, i.e., a "free energy" to minimize, the lowest-energy state is preferred and the dropletlike structure is unstable, just as in the phase-transition problem. We have shown recently by direct numerical integration of a model equation that nonvariational effects, i.e., the nonexistence of a Lyapunov functional, can stabilize dropletlike structures in the vicinity of a subcritical instability, and pointed out the similarity with solitons in conservative systems.[6] We show in this Letter that these localized structures can be computed perturbatively in slightly dissipative systems, and that *the leading-order effect of the dissipative terms is just to select the size of the structure* among a family of scale-invariant solitons. Although this selection can, in principle, occur in the vicinity of a supercritical bifurcation, the *stability* of the localized structure requires a subcritical bifurcation.

We consider a subcritical Hopf bifurcation, with a complex amplitude $W(x,t)$ governed by the Ginzburg-Landau equation,

$$\frac{\partial W}{\partial t} = \mu W + (\alpha + i)\frac{\partial^2 W}{\partial x^2} + (\beta + 2i)|W|^2 W + (\gamma + i\delta)|W|^4 W, \qquad (1)$$

where μ is the distance from criticality, α is positive, and β, γ, and δ are real. We have simplified the imaginary coefficients by appropriate scalings of W and space. Small perturbations are amplified when $\mu > 0$; the bifurcation is supercritical if $\beta < 0$ and subcritical for $\beta > 0$; in the subcritical case, $\gamma < 0$ is required for stability. Let us also recall that the Ginzburg-Landau equation with only real coefficients can be put in a variational form. This is not true for complex coefficients; i.e., the right-hand side of Eq. (1) is not proportional to the derivative of a functional with respect to $\overline{W}$ (the complex conjugate of W). Equation (1) can be derived for two-dimensional disturbances of the plane Poiseuille flow,[7] where W represents the complex amplitude of Tollmien-Schlichting waves. The traveling waves observed in binary-fluid-mixtures convection also take place via a subcritical Hopf bifurcation, but the right- and left-traveling waves $W_- \exp[i(\omega t - kx)]$ and $W_+ \exp[i(\omega t + kx)]$ must both be considered, and the amplitude equations for W_- and W_+ are coupled. We have numerically observed localized stationary solutions in both cases,[8] but for simplicity we will consider here the simplest model (1).

We have numerically integrated Eq. (1) with a pseudospectral method involving 512 complex modes and periodic boundary conditions on the interval $[0,L]$. A typical pulselike solution is shown in Fig. 1. It corresponds to a small region in the bifurcated state surrounded by the basic state. Notice that the amplitude of the pulse is strongly localized while its phase varies almost linearly in space. Solutions with a similar shape have been observed on large interval ranges of the constants α,β. Their typical size does not depend on the box length L which has been varied from 4π to 30π. The

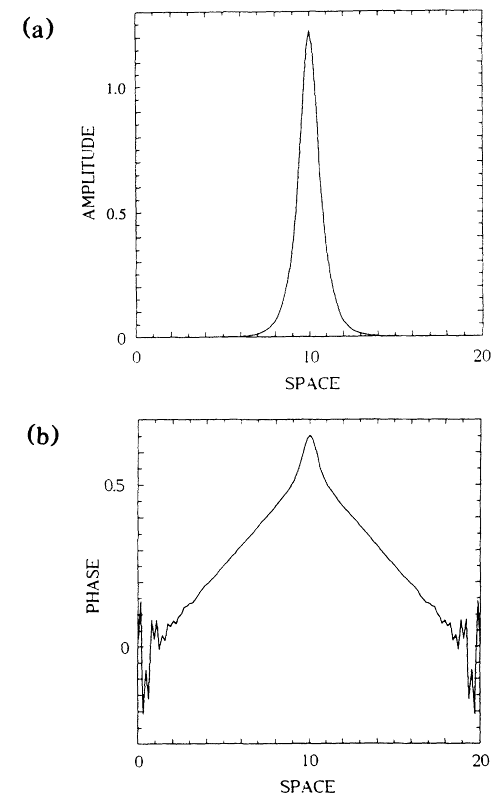

FIG. 1. 1D pulselike solution in the case $\mu=-0.24$, $\alpha=0.15$, $\beta=2.4$, $\gamma=-1.65$, and $\delta=2$; interval length, $L=20$. (a) Amplitude profile $|W(x)|$. (b) Phase profile $\phi(x)=\arg W(x)$. Although the coefficients of the dissipative terms are not small, the amplitude predicted by Eq. (7) agrees within 5% with the numerical result; however, the dissipative pulse size is about 65% the one of the corresponding soliton.

pulses exist for values of μ within a finite band. They are obtained with a great variety of initial conditions. For instance, a phase-unstable homogeneous state $|W|\neq 0$ often evolves to a pulselike solution.[9] Stationary localized pulses are thus structurally stable solutions of Eq. (1).

The shape of the pulse amplitude reminds one of the pulselike soliton of the nonlinear Schrödinger equation, which corresponds to $\mu=\alpha=\beta=\gamma=\delta=0$ in Eq. (1). We thus look for a perturbative approach with $\mu,\alpha,\beta,\gamma,\delta$ of order $\epsilon \ll 1$, and write Eq. (1) in the form

$$\frac{\partial W}{\partial t}=i\frac{\partial^2 W}{\partial x^2}+2i|W|^2W+\epsilon R(W). \quad (2)$$

When $R=0$ Eq. (2) has a well-known family of one-soliton solutions, $W_0(x,t)=r_0(x)\exp[-i\Theta_0(t)]$, with $r_0(x)=2\Delta_0\operatorname{sech}(2\Delta_0 x)$ and $\Theta_0(t)=-4\Delta_0^2 t+\Psi_0$. The existence of this family traces back to the invariance of the nonlinear Schrödinger equation under rotations in the complex plane, $W\to W\exp(i\theta)$, and dilatations $W\to\lambda W$, $x\to\lambda x$, and $t\to\lambda^2 t$.

Writing

$$W(x,t)=[r_0(x)+w(x,t)]\exp[-i\Theta_0(t)],$$

we get from Eq. (2) the linearized evolution equation for the perturbation w,

$$\partial_t w=Lw+\epsilon R(W)\exp(i\Theta_0), \quad (3)$$

with $Lw=i[-4\Delta_0^2 w+\partial_{xx}w+2r_0^2(2w+\bar{w})]$. The phase and dilatation invariances imply the existence of neutral eigenmodes for the operator L. One can check that they are, respectively, ir_0 and $\partial r_0/\partial\Delta_0$. Indeed, we have

$$L(ir_0)=0,\quad L(\partial r_0/\partial\Delta_0)=8\Delta_0(ir_0). \quad (4)$$

Equations (4) represent a codimension-two singularity.[10] In other words, they show that the size of the pulse and its phase are coupled (see the equations given in Ref. 11).

When $R\neq 0$ we thus look for slowly varying solitons in the form

$$W(x,t)=2\Delta(t)\operatorname{sech}[2\Delta(t)x]\exp[-i\Theta(t)]. \quad (5)$$

The temporal evolution of a soliton under the action of a perturbation is a well-known problem of soliton theory, and can be solved with the inverse-scattering method.[11] The temporal evolution of $\Delta(t)$ can be found in a simpler way here: Multiplying Eq. (1) by $\bar{W}$ and integrating on space leads to the evolution equation for $\int_{-\infty}^{\infty}dx\,|W|^2$,

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}dx\,|W|^2=\int_{-\infty}^{\infty}dx\left[\mu|W|^2-\alpha\left|\frac{\partial W}{\partial x}\right|^2+\beta|W|^4+\gamma|W|^6\right]. \quad (6)$$

We get from Eqs. (5) and (6) the evolution equation for the soliton size Δ,

$$\tfrac{1}{2}\,d\Delta/dt=\mu\Delta+\tfrac{4}{3}(-\alpha+2\beta)\Delta^3+\tfrac{128}{15}\gamma\Delta^5. \quad (7)$$

If $2\beta>\alpha$, Eq. (7) has two nonzero solutions $\Delta_\pm$ for $\mu_s<\mu<0$, with $\mu_s=5(-\alpha+2\beta)^2/96\gamma$. Only the larger Δ_+ is stable, and gives the size of the pulse. The dissipative terms of Eq. (1) thus stabilize one of the soliton solutions (5) and select its size by breaking the scale invariance associated with the conservative problem. Note that this selection mechanism among the family of solutions (5) by the dissipative terms of Eq. (1) also occurs when Eq. (1) represents a supercritical bifurcation ($\beta<0$). However, the existence of a stationary solution Δ in Eq. (7) then requires $\mu>0$, and the $W=0$ solution is no more stable. Thus, the stable pulselike solutions described here require the existence of a subcritical bifurcation. Then, in the interval range of μ where two stable homogeneous solutions of Eq. (1) exist, $\mu_c<\mu<0$, with $\mu_c=\beta^2/4\gamma$, there exists an order ϵ interval, $\mu_s<\mu<0$, where a pulselike solution of size

$\Delta_+(\mu,\alpha,\beta,\gamma)$ is selected. (Note that $\mu_c < \mu_s < 0$ because $2\beta > \alpha > 0$.)

The above mechanism also applies in the case of localized surface waves observed on a horizontal layer of fluid submitted to vertical vibrations.[2] If one considers only standing waves, the wave's complex amplitude obeys the equation[12]

$$\frac{\partial W}{\partial t} = (-\lambda + i\nu)W + \mu\overline{W} + i\frac{\partial^2 W}{\partial x^2} + 2i\,|W|^2 W, \quad (8)$$

where λ is the dissipation, $\lambda > 0$, ν is the frequency detuning from parametric resonance, μ is proportional to the external forcing, and the imaginary coefficients have been simplified by appropriate scalings of space and W. Equation (8) has exact pulselike solutions;[12] it is, however, interesting to note that the perturbative method used above shows again that two solitary waves of the form given by Eq. (5) are selected by the dissipative terms when the bifurcation described by Eq. (8) is subcritical, i.e., when $\nu < 0$. We find

$$d\Delta/dt = 2\Delta(-\lambda + \mu\cos 2\Theta),$$

$$d\Theta/dt = -4\Delta^2 - \nu - \mu\sin 2\Theta.$$

The phase $\Theta_\pm$ of the stationary solutions is quenched by the external forcing, $\Theta_\pm = \pm\frac{1}{2}\cos^{-1}(\lambda/\mu)$, and the size $\Delta_\pm$ of the wave envelope is given by $4\Delta_\pm^2 = -\nu \pm (\mu^2 - \lambda^2)^{1/2}$. As above, only the pulse of larger size is stable.

We have thus described a simple mechanism to explain the existence of stable localized structures in the vicinity of subcritical bifurcations. Other examples with applications to various experimental situations can be easily considered, and in particular the stability of multipulse solutions, observed in Eq. (1) can be investigated. Let us recall that these localized structures are unstable when a Lyapunov functional exists,[6] or marginally stable in the conservative case, i.e., when the coefficients of Eq. (1) are pure imaginary. The stabilization mechanism is a nonvariational effect that traces back to the coupling between the amplitude and the phase of the wave's complex amplitude. As reported in Ref. 6, this stabilization mechanism also works for two-dimensional fields $W(x,y,t)$, solutions of Eq. (1) where the diffusion term is the two-dimensional Laplacian; although in the nonlinear Schrödinger equation limit considered here, these two-dimensional pulselike solutions are known to be strongly unstable.

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[11] See, for instance, G. L. Lamb, *Elements of Soliton Theory* (Wiley, New York, 1980). The one-soliton solutions of the nonlinear Schrödinger equation evolve under the action of a perturbation ϵR, according to the equations

$$\frac{d\Delta}{dt} = \frac{1}{2}\,\mathrm{Re}\int_{-\infty}^{\infty} \epsilon R(W)\overline{W}\,dx,$$

$$\frac{d\Theta}{dt} = -4\Delta^2 - \frac{1}{2\Delta}\,\mathrm{Im}\int_{-\infty}^{\infty} \epsilon R(W)\overline{W}_\Delta\,dx,$$

with

$$W_\Delta = 2\Delta\,\mathrm{sech}[2\Delta(t)x] \times \{1 - 2\Delta(t)x\tanh[2\Delta(t)x]\}\exp[-i\Theta(t)].$$

The first equation gives the result (7) in the case of our model (1).

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