Phase-Space Constraints on Bosonic and Fermionic Dark Matter

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The widely used Tremaine-Gunn phase-space constraint giving lower-mass limits for neutrino dark matter is extended to bosons. The occupation number for a boson gas does not in general have a maximum, but only a small fraction of primordial bosons in equilibrium have large occupation numbers. Dark galaxy halos formed dissipationlessly from primordial bosons therefore have fairly low coarse-grained phase-space number densities. This leads to mass lower limits in the eV range for bosonic dark-matter candidates, only slightly smaller than for fermions.

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Elementary particles are likely candidates for explaining the dark matter in halos around galaxies and in galaxy clusters. Unfortunately their identity is not yet known, but major efforts have been spent on constraining their properties.

Among the most widely used methods for constraining the mass of dark-matter particles is the Tremaine-Gunn phase-space constraint[1] and variations thereof.[2] These methods work for fermions (e.g., neutrinos) that were once in thermal equilibrium with the primordial photon gas. Tremaine and Gunn noticed that the occupation number f, defined as $f(p)=h^3n(p)/g$, where $n(p)dp$ is the phase-space number density of particles with momenta between p and $p+dp$, h is Planck's constant, and g is the number of helicity states, had a maximum value of 0.5 for a primordial neutrino gas, and by Liouville's theorem there is also the maximum occupation number for any coarse-grained phase-space distribution created from primordial neutrinos by dissipationless processes.

The maximum value follows from the Fermi-Dirac distribution, which for zero chemical potential has occupation number $f_F(p)=1/[\exp(E/kT)+1]$, where the energy E is a function of momentum.[3] When particles decouple from thermal equilibrium, the occupation numbers have a maximum of 0.5. The full distribution of occupation numbers is shown in Fig. 1 for the two extreme limits: Fermions decoupling in the relativistic regime, $f_{\mathrm{FR}}(p)\approx 1/[\exp(pc/kT)+1]$, and fermions decoupling in the nonrelativistic regime, $f_{\mathrm{FN}}(p)\approx 1/[\exp(p^2/mkT)+1]$. The distribution $N(f)$ is simply the fraction of particles with an occupation number exceeding f.

The other set of curves shown in Fig. 1 are the corresponding distributions $N(\varphi)$ of coarse-grained occupation numbers, calculated from the "mixing-theorem" recipe given by Tremaine, Hénon, and Lynden-Bell.[4] $N(\varphi)$ is the maximal fraction of particles that can "be arranged" to have coarse-grained phase-space densities exceeding φ via dissipationless processes. The distribution is constructed by calculating the mass and phase-space volume, $M(f)$ and $V(f)$, of regions with fine-grained occupation exceeding f, and defining φ as $h^3M(f)/gmV(f)$. $N(\varphi)$ is then simply given as the corresponding value of $N(f)$. One notices, for instance, that 7.5% of primordial neutrinos decoupling in the relativistic regime have occupation numbers exceeding 0.25, whereas 15% of the coarse-grained distribution can have $\varphi>0.25$.[5] The similar fractions for nonrelativistic decoupling are 39% and 66%, respectively.

Whereas $N(\varphi)$, in general, exceeds $N(f)$, they both share 0.5 as the maximal occupation number. This is the basis of the Tremaine-Gunn argument, which in its simplest version says that the coarse-grained phase-space density of halo neutrinos cannot exceed the maximum value of the original fine-grained density. Assuming the dark-matter distribution to be an isothermal sphere with core radius $r_c=(9\sigma^2/4\pi G\rho_0)^{1/2}$, where ρ_0 is the central density, and σ is the Maxwellian one-dimensional velocity dispersion, the corresponding maximum phase-space density is $\rho_0 m_F^{-4}(2\pi\sigma^2)^{-3/2}$, where m_F is the fermion mass. Requiring this maximum to be less than g/h^3 (a factor of 2 included to account for equal numbers of particles *and* antiparticles) leads to[6]

$$m_F>\left[\frac{9h^3}{2(2\pi)^{5/2}gG\sigma r_c^2}\right]^{1/4}=(38\ \mathrm{eV})\sigma_{100}^{-1/4}r_{10}^{-1/2}g^{-1/4}. \quad (1)$$

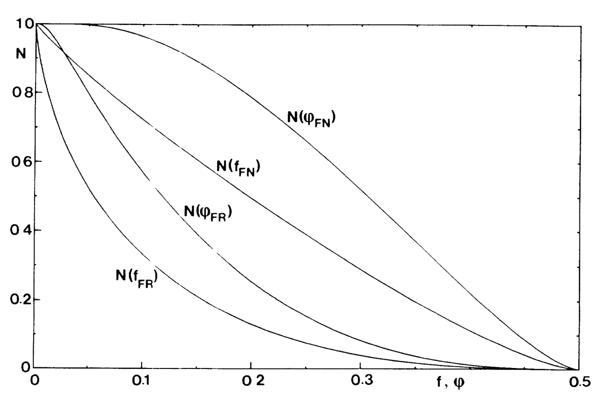

FIG. 1. Fine- and coarse-grained distributions of occupation numbers $N(f)$ and $N(\varphi)$ for fermions with zero chemical potential decoupling while relativistic (lower curves) and nonrelativistic (upper curves).

Here $\sigma_{100}=\sigma/100\ \mathrm{km\,s^{-1}}$ and $r_{10}=r_c/10$ kpc.

The mass lower limits obtained from Eq. (1) for observed galaxies tend to be close to the upper experimental bounds on the mass of the electron neutrino. Therefore, the precision of this equation has been extensively discussed.[2] For example, the choice of an isothermal sphere for the coarse-grained dark-matter distribution leads to overly restrictive limits on the fermion mass, when compared to other possible distribution functions, and even assuming the isothermal sphere, one must also be careful when choosing the parameters for observed galaxies, in particular the core radius, which is likely to be larger for the dark-matter distribution than for the observed stellar distribution.

Contrary to fermions, the Bose-Einstein distribution has no maximum occupation number when the chemical potential equals zero (we shall return to nonzero chemical potentials in a moment). The boson occupation number $f_B=1/[\exp(E/kT)-1]$ clearly diverges for small energies. Therefore, it has hitherto been assumed (as originally stated in Ref. 1) that phase-space arguments would not lead to a relation like Eq. (1) in the case of bosons. However, as will now be demonstrated, it is possible to derive a mass limit very similar to the one for fermions.[7]

Figure 2 shows the distributions $N(f)$ and $N(\varphi)$ for bosons decoupling while relativistic $(f_{\mathrm{BR}}(p)\approx 1/[\exp(pc/kT)-1])$ and decoupling in the nonrelativistic regime $(f_{\mathrm{BN}}(p)\approx 1/[\exp(p^2/2mkT)-1])$, respectively. While the occupation numbers have no upper limits, the crucial thing to notice is that the fraction of particles having large occupation numbers is rather small. For instance, less than 10% of bosons decoupling in the relativistic regime have coarse-grained phase-space density exceeding $2g/h^3$, and only 1% exceeds $6g/h^3$. For nonrelativistic decoupling the occupation numbers corresponding to 10% and 1% are of order 2×10^2 and 2×10^4, respectively.

Dissipationless formation of a dark-matter halo from primordial bosons would, from a phase-space point of view, be most efficient if particles were preferentially taken from the high-density part of phase space, and as these particles have the lowest momenta, they are likely to take part in halo formation. However, since the dark matter in halos is expected to incorporate at least 10% (perhaps more) of the total dark-matter abundance in the Universe, a significant fraction of the particles must necessarily have "low" occupation numbers. For relativistic decoupling, the "typical" bosons would have occupation numbers of a few. (The mean value of the occupation number for relativistic decoupling is 2.4 for the densest 10% fraction. For nonrelativistic decoupling the mean value diverges, but the median value for the densest 10% is 3×10^2. We shall later argue that the nonrelativistic case is probably not very relevant.) Taking this into account, one can derive a relation similar to Eq. (1) by comparing the assumed coarse-grained distribution (e.g., an isothermal sphere) to the "typical" fine-grained density, parametrized by $\tau g/h^3$ (an extra factor of 2 may be included to add the antiparticles if they are distinct from the particles). This results in a mass limit for bosonic dark matter:

$$m_B > \left[\frac{9h^3}{2(2\pi)^{5/2}g\tau G\sigma r_c^2}\right]^{1/4} = (38\ \mathrm{eV})\sigma_{100}^{-1/4}r_{10}^{-1/2}g^{-1/4}\tau^{-1/4}. \quad (2)$$

Since the mass limit depends on τ only through the fourth root, the limits for boson masses are only slightly smaller than the corresponding fermion mass limits for any given galaxy; the exact difference depends on details of the formation process.

One should bear in mind that in spite of the similarity between Eqs. (1) and (2), their logical foundations differ. The fermion limit is "firm" in the sense that it can be applied to any individual galaxy if one is willing to accept the assumed coarse-grained dark-matter distribution. The decisive point is the existence of a maximum for f_F. The boson limit is only correct in a statistical sense, i.e., it should be applied to a large sample of observed galaxies. The reason for this difference is the lack of a maximum for f_B. One could consider picking high-density bosons from a large volume of the Universe to make a single halo, thus getting a spuriously high value of τ for that particular galaxy, but one would get in trouble trying to do the same trick for a large number of galaxies, and might run out of high-density particles.

It has so far been assumed that the boson chemical potential μ is zero. A nonzero chemical potential must be negative in order to avoid negative values for the occupation numbers. Such a negative potential poses a problem because the occupation number now has a maximum, namely, $1/[\exp(|\mu|/kT)-1]$, calculated at the decoupling temperature. Equation (2) can now be firmly used choosing τ as the maximum occupation number, and lower values for τ may be chosen from considerations of the specific $N(\varphi)$ distribution, which has to be calculated

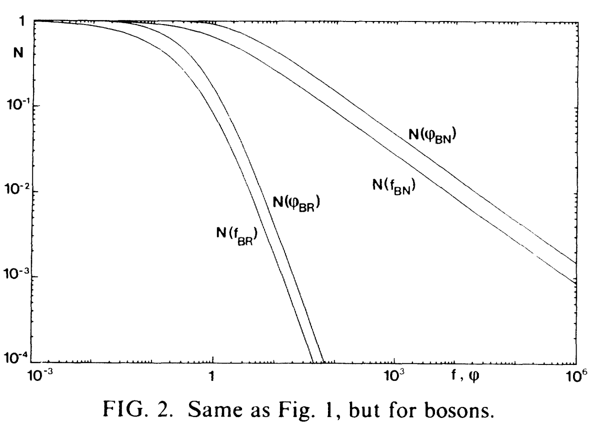

FIG. 2. Same as Fig. 1, but for bosons.

separately for each choice of μ/kT. For low values of $|\mu|/kT$, mass limits obtained in this manner closely resemble limits for a zero chemical potential.[8]

It is not within the scope of this presentation to enter into a detailed discussion of the possible existence of low-mass bosons. No such particles are known to exist, and (apart from the axion) there are no obvious theoretical reasons to expect their existence. On the other hand, there seems to be no argument against them either, as long as they do not couple too strongly to, for instance, photons. Should they exist, a weak coupling would indicate that they decoupled while they were ultrarelativistic, thus making $f_{\rm BR}$ the most likely original phase-space distribution. It is worth stressing that the limits derived in this Letter are applicable only to particles that were once in thermal equilibrium, such as thermal axions.[9] They do not apply to the nonthermal coherent axions,[10] which are formed in a Bose condensate, or to relic axions from decaying axionic strings.[11]

The main purpose of this Letter has been to show that the phase-space constraints previously used to place mass lower limits on fermionic dark-matter candidates can be extended to bosons. For a negative chemical potential a conservative value of τ in Eq. (2) can be calculated exactly, whereas one has to study the distribution of occupation numbers and rely on statistics for a sample of galaxies in the case of zero chemical potential. The equations given above for fermions and bosons can only be considered to give order-of-magnitude mass limits because of the specific choice of an isothermal sphere for the dark-matter distribution. One may construct other coarse-grained distributions that utilize the particles more efficiently, so as to lower the mass limits somewhat.[2] On the other hand, one can argue that nature was probably less efficient than theoretical astrophysicists in this respect, so that the "true" mass limits would tend to increase.

[1]S. Tremaine and J. E. Gunn, Phys. Rev. Lett. **42**, 407 (1979).

[2]See, for example, J. Madsen and R. I. Epstein, Astrophys. J. **282**, 11 (1984); Phys. Rev. Lett. **54**, 2720 (1985); R. Cowsik and P. Ghosh, Astrophys. J. **317**, 26 (1987); J. P. Ralston, Phys. Rev. Lett. **63**, 1038 (1989); G. Lake, Astron. J. **98**, 1253 (1989).

[3]Energies and chemical potentials do not include the rest-mass energy.

[4]S. Tremaine, M. Hénon, and D. Lynden-Bell, Mon. Not. Roy. Astron. Soc. **219**, 285 (1986).

[5]J. Madsen and R. I. Epstein [Astrophys. J. **282**, 11 (1984)] did not realize the distinction between $N(f)$ and $N(\varphi)$. Thus the caption to their Fig. 1 is incorrect, and so are the last two paragraphs in Sec. II. These matters were clarified in Ref. 4.

[6]This expression is identical to Eq. (6) in Ref. 1, except for the fact that Tremaine and Gunn include two equal-mass neutrino flavors, where Eq. (1) includes only one. Also, there is a misprint in Ref. 1 in that m_ν should scale as $r_c^{-1/2}$ instead of $r_c^{-1/4}$.

[7]After this work was submitted, Georg Raffelt called my attention to a paper by M. S. Turner [Phys. Rev. Lett. **59**, 2489 (1987)] on thermal production of axions, where a Tremaine-Gunn-type argument is applied to estimate the order-of-magnitude contribution of relativistically decoupling axions to the halo-density around galaxies. A detailed treatment of the bosonic phase-space distributions and consequences for dark-matter mass lower limits is, however, not attempted in that paper.

[8]For distinct bosons and antibosons annihilating into photons, non-negative occupation numbers for *both* particles and antiparticles require $|\mu| \leq 2m_B c^2$.

[9]Turner, Ref. 7; M. Khlopov (private communication).

[10]J. Preskill, M. B. Wise, and F. Wilczek, Phys. Lett. **120B**, 127 (1983); L. F. Abbott and P. Sikivie, *ibid.* **120B**, 133 (1983); M. Dine and W. Fischler, *ibid.* **120B**, 137 (1983).

[11]R. L. Davis, Phys. Lett. **180B**, 225 (1986).