Spacetime Singularities in String Theory

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It is shown that a large class of time-dependent solutions to Einstein's equation are classical solutions to string theory. These include metrics with large curvature and some with spacetime singularities. Unlike the case of orbifold singularities, it is shown that string propagation through the singular region is not well behaved.

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It has been known since the development of general relativity that certain symmetric exact solutions to Einstein's equation are singular. However, it was not until the late 1960s that Hawking and Penrose showed how pervasive these singularities are by proving their powerful singularity theorems.[1] One expects that the region near these singularities will not be adequately described by general relativity, and that one will require a more complete quantum theory of gravity. String theory is a promising candidate for such a theory. It modifies general relativity even classically at a scale α' set by the string tension, which is believed to be of the order of the Planck scale. The classical equation of motion for the metric in string theory is the condition for conformal invariance of a two-dimensional nonlinear σ model. It can be expressed in terms of σ-model perturbation theory as[2,3]

$$0=R_{\mu\nu}+\tfrac{1}{2}\alpha' R_{\mu\rho\sigma\lambda}R_{\nu}^{\rho\sigma\lambda}+\cdots, \tag{1}$$

where the ellipsis denotes terms constructed from derivatives and higher powers of the curvature. Clearly, Ricci flat metrics with curvature small compared to the Planck curvature are approximate solutions to (1). In this case, one can treat the higher-order terms as a small perturbation and obtain solutions which are qualitatively similar to those of general relativity. However, in the region near a singularity where the curvature is of the order of the Planck scale, the solutions to (1) can deviate significantly from the predictions of general relativity. A fundamental question of string theory is whether exact solutions to (1) are also singular. If not, then the ubiquitous singularities of general relativity would be removed in string theory even at the classical level without invoking quantum mechanics.

In order to answer this question, one might first try to apply the singularity theorems of general relativity to string theory. These theorems do not require the detailed form of Einstein's equation, but they do assume an energy condition which, roughly speaking, says that gravity is always attractive. Unfortunately, the higher-order terms in (1) violate this condition, and so the theorems are not applicable.

In the absence of a general theorem, the next step is to look for examples. Most known solutions to string theory with strong curvature take the form of a product of Minkowski space and a compact manifold. These static spacetimes do not shed any light on the dynamics of string theory in the strong-curvature regime or on the existence of singularities.[4] One exception has been discussed recently. If one takes flat spacetime and adds a dilaton field which is a linear function of the time coordinate, then one has an exact solution to the string equations of motion.[5] If one now rescales the flat metric by a function of the dilaton so that the new metric is governed by the standard Einstein action, then the solution corresponds to a linearly expanding Robertson-Walker cosmology.[6] In particular, there is a curvature singularity at $t=0$. However, there is clearly something wrong with interpreting this as a singular solution to string theory. The new metric differs from the original flat metric by a field redefinition which should not change any physical predictions. In general relativity, singularities are defined in terms of the motion of test particles. In string theory it seems natural to focus on the motion of "test strings." For the above solution, however, strings couple directly to the original flat metric. So it does not seem reasonable to interpret this as a singular solution to string theory.

Amati and Klimcik[7] have recently shown that a large class of vacuum solutions in general relativity are classical solutions to string theory. In this Letter we first give a new derivation of this result which is more geometric. We also generalize these solutions to include the dilaton and antisymmetric tensor field, which are also massless fields of string theory. We then consider the question of singularities. Most of these solutions are singular in the sense of general relativity, i.e., they are geodesically incomplete. We will show that they are also singular in the sense of string theory. Namely, a string which tries to propagate through the singular region becomes infinitely excited. We conclude with a few remarks about singularities in quantum string theory.

We begin by discussing a class of solutions to Einstein's vacuum field equation. Consider metrics of the form

$$ds^2=-du\,dv+dx^i dx_i+F(u,x^i)du^2. \tag{2}$$

The coordinates have been chosen so that the surfaces

u=const (but not, in general, v=const) are null. x^1 and x^2 are the two transverse coordinates. (The following analysis can be trivially extended to D dimensions by letting i run from 1 to $D-2$.) The metrics are independent of v so there is a null Killing vector l^μ. It is easy to check that l^μ is not only a Killing vector, but is actually covariantly constant: $\nabla_\mu l^\nu=0$. In terms of l^μ the metrics are simply $g_{\mu\nu}=\eta_{\mu\nu}+Fl_\mu l_\nu$, with inverse $g^{\mu\nu}=\eta^{\mu\nu}-Fl^\mu l^\nu$.

The curvature of these metrics is easily computed:

$$R_{\mu\nu\rho\sigma}=2l_{[\mu}\partial_{\nu]}\partial_{[\rho}F\,l_{\sigma]}\,. \tag{3}$$

(The right-hand side is covariant since the partial derivatives are equivalent to covariant derivatives in this expression.) It follows from Eq. (3) that the Riemann tensor is orthogonal to l^μ on all indices, which is consistent with the fact that l^μ is covariantly constant. The Ricci tensor is thus $R_{\mu\nu}=-\frac{1}{2}\partial^2F\,l_\mu l_\nu$. Since F is independent of v, $\partial^2F=\partial_T^2F$, where ∂_T^2 is the transverse Laplacian. Thus the only constraint on F in order for these metrics to be exact vacuum solutions to Einstein's equation[8] is that F satisfy Laplace's equation for each u. Since the u dependence is arbitrary, one obtains solutions which depend on arbitrary functions.

One class of solutions is $F(u,x^i)=h_{ij}(u)x^ix^j$, where $h_{ij}(u)$ is symmetric and traceless. These are known as exact plane waves in general relativity. Although it is not obvious in the coordinates we are using, the plane-wave solutions have a five-parameter symmetry group which includes translations in the transverse directions along the wave front. Since one can choose the two components of h_{ij} arbitrarily for each Fourier component $e^{i\omega u}$, these solutions have exactly as much freedom as linearized gravitational waves. But by Eq. (3), the curvature can be arbitrarily large. These solutions have been extensively studied. It has been shown that they have no Cauchy surface,[9] and when coupled to quantum matter fields, they have no particle creation[10] or vacuum polarization.[10,11] More general solutions to Laplace's equation lead to generalizations of the plane waves which typically have no symmetries other than the one generated by l^μ. These spacetimes are known as plane-fronted waves.

Amati and Klimcik[7] have recently shown that plane-fronted waves are solutions to Eq. (1) to all orders in σ-model perturbation theory. Their approach was to directly study the σ-model divergences. We now give a more geometrical derivation of this result based on the fact that the curvature (3) is null. The higher-order terms in the equation of motion (1) are all second-rank tensors constructed from powers of the Riemann tensor and its derivatives. The only possible term involving just one Riemann tensor is $\nabla^\mu\nabla^\nu R_{\mu\alpha\nu\beta}$. But this vanishes since it can be reexpressed solely in terms of the Ricci tensor by the Bianchi identity. From Eq. (3), any expression with more than one Riemann tensor must contain more than two l_μ's. Since there are only two free indices, one of the l_μ's must be contracted. If l_μ is contracted on any index of the Riemann tensor, then the term vanishes since l_μ is covariantly constant and orthogonal to $R_{\mu\nu\rho\sigma}$ on all indices. If l_μ is contracted on a covariant derivative, then it also vanishes since

$$l^\mu\nabla_\nu\nabla\cdots\nabla R=\mathcal{L}_l(\nabla\cdots\nabla R)=0\,, \tag{4}$$

where $\mathcal{L}$ denotes the Lie derivative, and we have used the fact that l^μ is a covariantly constant Killing vector. Since all contractions of l_μ vanish, *all higher-order terms in the string equation of motion are automatically zero.*

String theory also imposes a constraint on the dimension of spacetime. This takes the form of $D=10$ (or 26 for the bosonic string) plus a scalar function of the curvature and its derivatives. Since we have just argued that all such terms vanish, the critical dimension is unchanged from its flat-space value. To satisfy this constraint, one can either consider x^i to run over all 8 (or 24) transverse dimensions or take the product of the four-dimensional solutions (2) with a time-independent internal conformal field theory.

One advantage of the above geometrical argument over a direct examination of the σ model is that it applies not only to string theory but to a wide class of theories. One has the following general result. *Any theory containing a spacetime metric whose field equation can be expressed in terms of a second-rank tensor constructed from the curvature and its derivatives has the plane-fronted waves as solutions.* In particular, any theory of gravity coupled to massive fields has a low-energy effective action of this form when the massive fields are integrated out.

To investigate the possibility of nonperturbative contributions to the equations of motion, one must study the σ model directly. In Ref. 7 it is shown that exact plane waves are exact solutions to string theory, even nonperturbatively. In other words, they correspond to exact conformal field theories. This provides a new class of conformal field theories which depend on arbitrary functions and describe string propagation in time-dependent backgrounds. Most of the more general plane-fronted waves are topologically trivial. In these cases, world-sheet instantons should not exist, and we therefore expect that they are solutions nonperturbatively as well.

It is easy to extend the plane-fronted waves to include the other bosonic massless fields of the string. Let the metric be of the form (2), the dilaton Φ be an arbitrary function of u, and the antisymmetric tensor be of the form

$$H_{\mu\nu\rho}=A_{ij}(u)l_{[\mu}\nabla_\nu x^i\nabla_{\rho]}x^j\,. \tag{5}$$

Then to leading order in σ-model perturbation theory, the field equations[3] for Φ and $H_{\mu\nu\rho}$ are automatically satisfied. The equation for the metric is

$$R_{\mu\nu}-\tfrac{1}{4}H_{\mu\rho\sigma}H_{\nu}^{\rho\sigma}-\nabla_{\mu}\nabla_{\nu}\Phi=0\,. \tag{6}$$

Since all terms in this equation are proportional to $l_\mu l_\nu$, it is satisfied provided that

$$\partial_I^2 F+\tfrac{1}{18}A_{ij}A^{ij}+2\Phi''=0\,. \tag{7}$$

So one can choose $A_{ij}(u)$ and $\Phi(u)$ arbitrarily, and solve (7) for $F(u,x^i)$. Since we can still add to F arbitrary solutions of the homogeneous equation, there is clearly a large class of solutions.

In σ-model perturbation theory, the leading-order equations include all terms with two derivatives of Φ and two $H_{\mu\nu\rho}$'s. We now consider the higher-order corrections. They are again of the form of second-rank tensors and scalars constructed from powers of $\nabla_\mu\Phi$, $H_{\mu\nu\rho}$, the metric, and their derivatives. Since $\nabla_\mu\Phi$ is proportional to l_μ, at most two derivatives of Φ can appear in any second-rank tensor. Since $H_{\mu\nu\rho}$ contains one l_μ and $H_{\mu\nu\rho}l^\rho=0$, the argument given earlier for the Riemann tensor shows that all terms constructed from more than two $H_{\mu\nu\rho}$'s and their derivatives vanish. Similarly, any term containing at least one Riemann tensor and one or more $\nabla_\mu\Phi$'s or $H_{\mu\nu\rho}$'s also vanishes. The only remaining candidates for higher-order terms are $(\nabla\cdots\nabla H)^2$. However, one can show that $l_{[\mu}\nabla_\nu x^i\nabla_{\rho]}x^j$ is covariantly constant. (In four dimensions, this follows from the fact that it is proportional to $\epsilon_{\mu\nu\rho\sigma}l^\sigma$, but it is true in any number of transverse dimensions.) Thus all derivatives of $H_{\mu\nu\rho}$ act only on $A_{ij}(u)$ which results in more l_μ's, and so the term vanishes. Hence fields satisfying (7) are also solutions to string theory to all orders in σ-model perturbation theory. It seems likely that the methods of Ref. 7 can be adapted (at least when F corresponds to an exact plane wave) to show that these backgrounds are solutions nonperturbatively as well.

We now turn to the question of singularities. For simplicity, we set the dilaton and antisymmetric tensor field to zero. As is often the case for solutions to Einstein's equation, most plane-fronted waves are singular. The simplest way to see this is to introduce a complex coordinate $z=x^1+ix^2$ on the transverse planes. The general plane-fronted wave can then be described by $F(u,x^i)=f(u,z)+\bar{f}(u,\bar{z})$. One can now distinguish three classes of singular solutions. The first is when f grows faster than z^2 for large z. In this case the curvature is unbounded at infinity. The second is when f diverges at some finite value of z, having, e.g., a pole or essential singularity. The third class corresponds to functions f which diverge at some finite value of u. In fact, the only nontrivial plane-fronted waves which are not singular are the exact plane waves $f(u,z)=g(u)z^2$ for smooth g.

All three classes of solutions are singular in the sense of having incomplete geodesics. This is the standard criterion for a singular spacetime in general relativity. One can also show that they have curvature singularities (rather than, e.g., conical singularities) since some components of the Riemann tensor in a parallelly propagated frame diverge along the incomplete geodesic. One should note that the strength of the singularity, i.e., the rate at which the curvature diverges, can be made arbitrarily high.

However, the question of most interest for string theory is whether these spacetimes admit consistent string propagation through the singular region. If one considers a purely classical string, then the answer is clearly no since geodesics are also solutions to the string equation of motion when there are no internal excitations. It turns out that even a quantized string cannot propagate through these singularities. We illustrate this with one example here. A more complete discussion will appear elsewhere.[12] Plane-fronted waves are ideally suited for a light-cone gauge analysis. One can choose a time coordinate on the worldsheet such that $u=p\tau$ and solve the constraints to determine v in terms of the x^i. Before proceeding to our example, we mention one special case that has already been discussed in the literature.[13] This is a shock wave $f(u,z)=\delta(u)h(z)$. A string in this background has the standard flat-space spectrum for $u<0$ and $u>0$ since the curvature vanishes in these regions. The interaction with the shock wave can cause transitions from one string state to another. By imposing suitable matching conditions across the shock, a formal expression relating in states to out states can be obtained. For the case when the center of mass z_0 at $\tau=0$ is far from the singularities of $h(z)$, this expression can be explicitly evaluated. One finds that if the string starts in an unexcited state, then after the shock wave passes, the expectation value of the number operator of the nth mode goes like $\langle N_n\rangle \sim |h''(z_0)|^2/n^2$. Although the total number of excitations is finite, $\sum\langle N_n\rangle<\infty$, the total mass diverges, $\langle M^2\rangle\sim\sum n\langle N_n\rangle=\infty$. One can interpret this result as evidence that strings cannot physically propagate through an infinite shock wave. (This does not seem to have been noticed by the authors of Ref. 13.)

The example we now consider has a much stronger singularity than a shock wave. Consider the case of a singular plane wave, $f=g(u)z^2$, where g diverges arbitrarily rapidly as $u\to 0$. This spacetime is analogous to one with a cosmological singularity. All timelike geodesics are incomplete. (By contrast, plane-fronted waves which are singular at some value of z are more analogous to spacetimes describing gravitational collapse. Some timelike geodesics are complete, but others are not.) From the standpoint of general relativity, the spacetime only exists for $u<0$. It does not make sense to ask if a string in this background can propagate through the singularity, since there is no spacetime for it to propagate to.

However, it seems reasonable to approach this problem in terms of a limiting procedure. Consider a oneparameter family of nonsingular plane waves (labeled by λ) such that $g(u,\lambda)=0$, i.e., the spacetime is flat, for $u<-u_0$ and $u>u_0$ and $\int_{-u_0}^{u_0}|g(u,\lambda)|^{1/2}du$ diverges as λ goes to infinity. One advantage of considering plane waves is that the equation for the transverse degrees of freedom is linear and so the different modes decouple. For each λ we investigate the behavior of a string propagating in this background and find that if $\langle M^2\rangle$ is finite initially, it is also finite at late times. But for a large class of functions $g(u,\lambda)$ of the above form, one can show[12] that for modes with $n^2 \ll \max|g(u,\lambda)|$,

$$\langle N_n\rangle \sim \exp\left[2\int_{-u_0}^{u_0}|g(u,\lambda)|^{1/2}du\right]\Big/n^2. \qquad (8)$$

So as $\lambda \to \infty$, not only does $\langle M^2\rangle$ diverge, but the excitation level of each mode of the string individually diverges. This shows that the limiting spacetime is indeed a singular solution to string theory.

One drawback of the singular solutions we have been discussing should be mentioned. They do not describe singularities which evolve from nonsingular initial conditions. This is a result of the null translational symmetry. It remains an important open question whether singularities can arise from nonsingular initial conditions in string theory.

We have shown that classical string theory admits solutions with spacetime singularities. Perturbative quantum corrections are unlikely to improve the situation. For the bosonic string, one-loop corrections can generate a cosmological constant. This adds a term proportional to the metric to the equations of motion which will modify the plane-fronted waves. However, for the superstring (possibly compactified on a supersymmetric internal space), this term is absent. As we mentioned earlier, massive fields will not cause any difficulty since their loop contributions can always be expanded in powers of derivatives. The result will again be polynomials in the curvature which will vanish for plane-fronted waves. *A priori*, massless fields may generate nonlocal terms which cannot be expanded in derivatives. However, one can show that plane-fronted waves admit a covariantly constant (two-component) spinor, which implies that they have unbroken supersymmetry. Some form of nonrenormalization theorem might then show that they are exact solutions to all orders in the string-loop expansion.

Little can be said at this time about the status of singularities in the full quantum string theory. If at least one phase of the full theory can be expressed in terms of a covariant equation in an invertible $\langle g_{\mu\nu}\rangle$ and its derivatives, then perhaps these solutions show the existence of singularities in the full theory. However, since quantum string theory probably involves qualitatively new effects such as spacetime topology change, this is unlikely to be the case. The final answer must await a more complete understanding of the nonperturbative theory.

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