Supersymmetric t-J Model in One Dimension: Separation of Spin and Charge

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Using the Bethe-*Ansatz* technique, we diagonalize exactly the one-dimensional t-J Hamiltonian for the supersymmetric case $T=J$. In this limit it is identical with models considered previously by Sutherland and by Schlottmann. The ground state is a liquid of singlet pairs of varying spatial separation for all band fillings. We find two types of gapless excitations with effective Fermi surfaces at $2k_F$ and k_F which we identify with the holon and the spinon excitations near half filling.

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The discovery of high-temperature superconductors has greatly stimulated the interest in strongly correlated systems. In particular, Anderson has suggested that the t-J model is an appropriate starting model.[1,2] The t-J model is characterized by a lattice Hamiltonian $\mathcal{H}$ which describes fermions with hard-core repulsion, nearest-neighbor hopping (t), and spin exchange (J). Considered in its own right, the model can be studied for any dimension and for all values of the ratio J/t. In this Letter, we consider one dimension and study the exact solution at $t=J>0$. We emphasize that the model we solve is not identical to the large-U limit of the repulsive Hubbard model,[3,4] which maps onto the limit $J \ll t$.

Using the Bethe-*Ansatz* method, this model was first solved by Sutherland[5] in a study of a multicomponent lattice gas. In particular, he derived the Bethe-*Ansatz* equations for the case of two fermions and one boson which reduces to the t-J model. A different form of the Bethe-*Ansatz* equations was discovered by Schlottmann,[6] who solved them for the ground state and discussed the thermodynamic properties of the model, applying the results to heavy-fermion systems. In addition, numerical calculations on finite clusters have been recently performed by Imada and Hatsugai[7] and by von Szczepanski *et al.*[8] In this Letter, we present for the first time a detailed analysis of the ground state and of the elementary-excitation spectrum at arbitrary filling.[9] We interpret the spectra in terms of solitonlike excitations which we identify as holons and spinons[10] near half filling. We show that doping the system with holes naturally leads to the separation of the spin and charge degrees of freedom.[10]

In the following we solve the Bethe equations for the ground state by means of a two-string *Ansatz* for the electron rapidities. The solution can be interpreted as a liquid of bound singlet pairs of varying spatial separation and binding energy. We solve for the elementary excitations of the model and show how the two branches involving charge and spin excitations can be interpreted as holons and spinons, respectively.

Consider a one-dimensional lattice of N_a sites with N electrons where each site is capable of accommodating at most one fermion. The dynamics is described by the t-J Hamiltonian

$$\mathcal{H} = -t\mathcal{P}\sum_{\langle i,j\rangle,\sigma} c_{i\sigma}^{\dagger}c_{j\sigma}\mathcal{P} + J\sum_{\langle i,j\rangle}(\mathbf{S}_i\cdot\mathbf{S}_j - n_i n_j/4)\,,$$

where the projector $\mathcal{P}=\prod_i(1-n_{i\uparrow}n_{i\downarrow})$ restricts the Hilbert space by the constraint of no double occupancy. The symmetries of this Hamiltonian are U(1) gauge, SU(2) spin, and lattice translational invariance. In addition, the model becomes supersymmetric at $t=J$.[11,12] The short-range nature of the interaction motivates the following *Ansatz*[13] for the amplitudes

$$\psi_{\sigma_1\cdots\sigma_N}(x_1,\ldots,x_N)$$

in the sector $x_{Q_1}<x_{Q_2}<\cdots<x_{Q_N}$:

$$\psi_{\sigma_1\cdots\sigma_N}(x_1,\ldots,x_N) = \sum_P(-1)^P A_{\sigma_{Q_1}\cdots\sigma_{Q_N}}(QP) \times\exp\left(i\sum_{j=1}^{N}k_{P_j}x_j\right).$$

P and Q denote permutations of $1,\ldots,N$, $(-1)^P$ is the sign of the permutation P, and we choose $x_i\neq x_j$ whenever $i\neq j$. The condition that $|\Psi\rangle$ be an eigenstate of $\mathcal{H}$ establishes a linear relation between the amplitudes $A_{\sigma_{Q_1}\cdots\sigma_{Q_N}}(QP)$. The multiparticle scattering matrix defined by these relations factorizes into a product of two-particle scattering matrices provided the Yang-Baxter equations are fulfilled.[13] Also, the Yang-Baxter equations represent the conditions for the consistency of the Bethe *Ansatz*[14] and require that $t=|J|$. Applying the quantum inverse-scattering method[15] we obtain a set of coupled algebraic equations[6,16] for the electron rapidities $\{v_j\}$, $j=1,\ldots,N$, and the spin rapidities $\{\Lambda_a\}$, $a=1,\ldots,M$ (M is the "number of down spins" in the solution):

$$\prod_{j=1}^{N}\frac{v_j-\Lambda_a+i/2}{v_j-\Lambda_a-i/2} = -\prod_{\beta=1}^{M}\frac{\Lambda_\beta-\Lambda_a+i}{\Lambda_\beta-\Lambda_a-i}\,,$$
$$\left(\frac{v_j+i/2}{v_j-i/2}\right)^{N_a} = \prod_{\beta=1}^{M}\frac{v_j-\Lambda_\beta+i/2}{v_j-\Lambda_\beta-i/2}\,, \qquad (1)$$

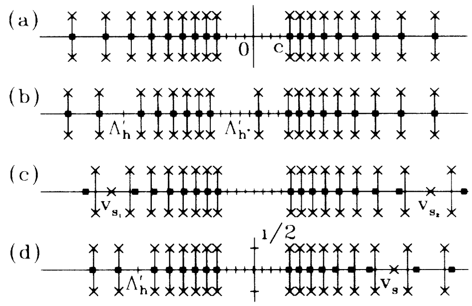

FIG. 1. Electron rapidities in the complex plane: Crosses denote the quantum numbers v_a' describing kinetic degrees of freedom, and solid squares denote the quantum numbers Λ_a' associated with spin degrees of freedom. (a) Ground state: Electron rapidities occur in complex pairs, $v_a'=\Lambda_a'\pm i/2$, describing singlet pairs of range $2/\ln(1+\Lambda_a'^{-2})$. The parameter c determines the filling factor N/N_a. (b) Holon-antiholon (h-h^*) excitation: A string at Λ_h' is transferred to a higher-energy state at Λ_{h^*}'. (c) Triplet (s-s) excitation: A string is broken up into two real rapidities v_{s_1} and v_{s_2}, each of which is describing a spinon. The two spinons combine into a triplet excitation as one of the spin rapidities Λ_a' has been removed. (d) Real-particle (s-h) excitation: Removing one electron leaves the system in an excited state with one holon at Λ_h' and one spinon at v_s. The holon and the spinon account for the charge and spin degrees of freedom, respectively, of the many-body state.

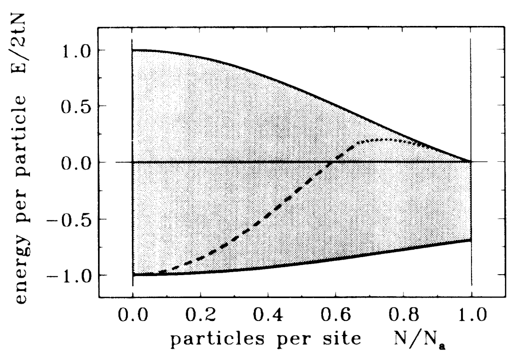

FIG. 2. Energy per particle $E/2tN$ vs filling factor N/N_a. The ground state is a liquid of singlet pairs of varying range described by complex pairs of rapidities; see Fig. 1(a). The highest accessible state is the ferromagnetic state with real rapidities v_a' and no spin rapidities Λ_a'. The dashed line denotes the lowest singlet state with real rapidities v_a'. This state is forced into a state with finite magnetization for a filling $N/N_a>2/3$ (dotted line). The overall width in energy decreases from the free-electron value $4t$ per particle for $N/N_a\to 0$ to $2t\ln 2$ in the Heisenberg limit $N/N_a\to 1$.

where $2v_j=\cot(k_j/2)$ for $t=J$. For even N the low-energy states are parametrized by a sea of two-strings in the complex plane with $v_a'=\Lambda_a'\pm i/2+O(e^{-N_a})$. Taking the logarithm of Eq. (1) we introduce the bare quantum numbers I_a', $a=1,\ldots,M=N/2$, which specify the roots of the equation. The I_a' are integers or half-odd integers and restricted to the interval $|I_a'|\le I_{\max}'=(N_a-M-1)/2$. For arbitrary filling the number of available quantum numbers I_a' exceeds the number of actual two-strings, so there is freedom in the choice of the set $\{I_a'\}$ to be occupied.

For the ground state, the $I_a'=I_a^{0\prime}$ are chosen symmetrically with respect to zero, $0\le I_{\min}'\le|I_a^{0\prime}|\le I_{\max}'$. The corresponding distribution of two-strings in the complex plane is shown in Fig. 1(a). In the thermodynamic limit ($N_a\to\infty$, $N/N_a=$const) we obtain an integral equation of Fredholm type for the distribution of the roots Λ_a'. At half filling, this integral can be solved in closed form and the ground-state energy of the Heisenberg chain, $E/N=-2t\ln 2$, is recovered.[6,17] Away from half filling, the integral equation has to be solved numerically: The corresponding ground-state energy is shown in Fig. 2. For all fillings N/N_a the ground state is a liquid of singlet bound pairs, where each singlet can be associated with a two-string at Λ_a'. The coherence length ξ of a particular pair depends on the position Λ_a' of the associated rapidity, $\xi=2/\ln(1+\Lambda_a'^{-2})$. In particular, the ground state involves pairs of arbitrarily weak binding energy ($|\Lambda_a'|\to\infty$) resulting in a gapless excitation spectrum. As in the *attractive* Hubbard model[18,19] the ground state of the t-J model is parametrized by complex pairs of rapidities. However, the physics is more like that of the *repulsive* Hubbard model. There is no jump in the chemical potential for adding one or two particles and we do not observe any transition as a function of filling N/N_a. Note that the repulsive Hubbard model involves real electron and spin rapidities. For the t-J model, real rapidities lead to an excited state as shown in Fig. 2.

There are two types of elementary excitations, which do not change particle number, involving (i) charge and (ii) spin degrees of freedom: (i) Charge excitations away from half filling involve the transfer of a particular bare quantum number $I_h'\in\{I_a^{0\prime}\}$ to a previously unoccupied state I_{h^*}' above the pseudo Fermi surface at $I_{\min}'$. This excitation transfers a charge e (not $2e$) into a higher-energy state but differs from the usual particle-hole excitation in a Fermi liquid as no spin is involved. According to Anderson's terminology for the strongly correlated Hubbard model,[10] we identify this excitation with a holon (kink of charge 1) and the corresponding spectrum with the holon-antiholon branch. The spectrum is obtained by solving the Bethe equations for the rapidities Λ_a' to order $1/N_a$. A shown in Fig. 1(b) the electron rapidities v_a' and the spin rapidities Λ_a' remain aligned. The holon-antiholon spectrum is shown in Fig. 3. $N/N_a=2/3$ marks the special filling (of high symmetry) above which a gap in momentum occurs, excluding excitations with momenta between $2\pi-3k_F$ and $3k_F$ ($k_F=\pi N/2N_a$). Keeping the antiholon fixed (e.g., $\Lambda_{h^*}'=c$) and moving the holon over the allowed parameter range, we obtain the holon excitation spectrum with an effective Fermi surface spanning $4k_F$. With respect

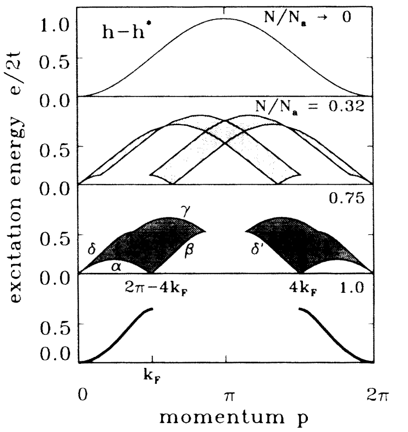

FIG. 3. Holon-antiholon (h-h^*) excitation spectrum for several values of the filling factor N/N_a. Starting with $\Lambda_h' = \Lambda_{h^*}' = c$ [see Fig. 1(b)], branch α is obtained by moving Λ_{h^*}' to $-c$, branch β is due to moving Λ_h' out to ∞, γ corresponds to moving Λ_{h^*}' back to c, and δ completes the loop as Λ_h' is moved back to c. The branches δ and δ' make up the holon excitation spectrum spanning a Fermi surface of range $4k_F$ ($k_F = \pi N/2N_a$). Note that a gap appears in momentum for filling $N/N_a < 2/3$, where the Hilbert space starts to shrink rapidly due to the constraint of no double occupancy.

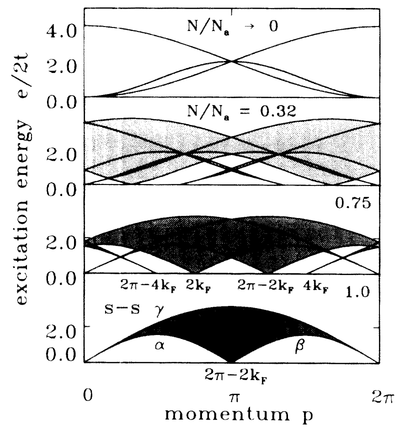

FIG. 4. Triplet (spinon-spinon, s-s) excitation spectrum for several values of the filling factor N/N_a. According to Ref. 16 a triplet excitation is two-parametric, one parameter describing a kink of spin $\frac{1}{2}$ or spinon. For $N/N_a \to 1$ the spectrum of the Heisenberg model is recovered: The branches α, β, and γ are obtained by starting with $v_{s_1} = v_{s_2} = \infty$ and moving $v_{s_1} \to -\infty$ (α), $v_{s_2} \to -\infty$ (β), and finally taking $v_{s_1} = v_{s_2}$ together back to ∞ (γ); see Fig. 1(c). The branch α is the spinon excitation spectrum spanning a Fermi surface of $2k_F$. The lowest (gapless) excitation is obtained by breaking a singlet pair at $\Lambda_a' = \pm\infty$, where the binding energy goes to zero. As $N/N_a \to 0$ the free-particle triplet excitation spectrum is recovered.

to Fig. 1(b) we identify the boundaries $\pm c$ with the pseudo Fermi surface for the holons.

(ii) The spin excitations, on the other hand, involve the breaking of a pair with (triplet) or without (singlet) spin flip. Here we restrict ourselves to the triplet excitations for the case of an even number of particles. The excitation consists in transferring a pair of complex roots onto the real axis and simultaneously removing the spin rapidity associated with the pair. As a consequence the remaining spin rapidities Λ_a' shift with respect to the electron rapidities v_a' as shown in Fig. 1(c). Again the excitation is two-parametric. At half filling the real rapidities v_{s_1} and v_{s_2} describe kinks of spin $\frac{1}{2}$ which combine into a triplet (or singlet) excitation as shown by Faddeev and Takhtajan[20] for the Heisenberg chain. The excitation spectrum shown in Fig. 4 is gapless for all fillings. This is due to the presence of asymptotically unbound pairs, i.e., pairs of arbitrarily weak binding energy. Real rapidities $|v_s| > c$ embedded in the sea of singlet pairs are identified with spinons as the corresponding excitation carries spin and no charge. On the other hand, isolated rapidities with $|v_s| < c$ are associated with real-particle excitations (carrying both spin and charge). Upon decreasing the filling N/N_a, the spinon excitation spectrum gradually transforms into a real-particle excitation spectrum as shown in Fig. 4. We find that the effective Fermi surface for the spinons is at k_F [corresponding to the points $v_s = \pm\infty$ in Fig. 1(c)].

Finally, the real-particle excitation spectrum involving a change in particle number is shown in Fig. 5. Removing a real particle near half filling leaves the system in an excited state [see Fig. 1(c)]: The hole splits into two solitonlike excitations, a spin-$\frac{1}{2}$ kink (spinon) and a spinless kink of charge e (holon). As in the strongly correlated Hubbard model, the excitation spectrum goes linearly to zero at $|k| = k_F$ and at $|k| = 3k_F$.

In conclusion, we have determined the ground state

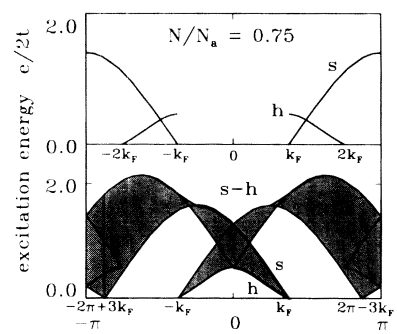

FIG. 5. Single-particle (s-h) excitation spectrum. Removing a particle leaves the system back in an excited state characterized by a spinon s and a holon h; see Fig. 1(d). Top: holon and spinon excitation spectra with Fermi surfaces at $2k_F$ and at k_F, respectively. Bottom: combination of the s and h excitation spectra into a real-particle (s-h) excitation spectrum. The state at k_F ($3k_F$) is a combination of a $2k_F$ holon and a $-k_F$ (k_F) spinon. The spectrum has been folded back into the first Brillouin zone.

and the elementary-excitation spectrum of the t-J model for arbitrary filling $N/N_a \leq 1$. We find that the repulsive on-site interaction dominates the attractive spin interaction J. We believe that the model belongs to the same universality class as the repulsive Hubbard model. Therefore we do not expect a phase transition in the interval $0 < J/t \leq 1$. This is consistent with the renormalized mean-field theory in one dimension by Zhang *et al.*[21] The gapless excitations at $2k_F$ (spin) and $4k_F$ (charge) produce long-range incommensurate spin and charge correlations in the t-J model.[22] Schulz has recently shown how to determine the asymptotic form of the correlation functions using results of the Bethe-*Ansatz* solution.[23]

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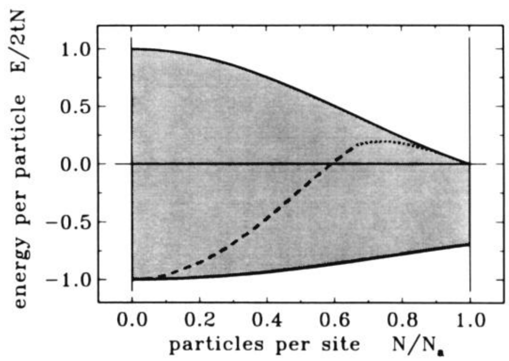

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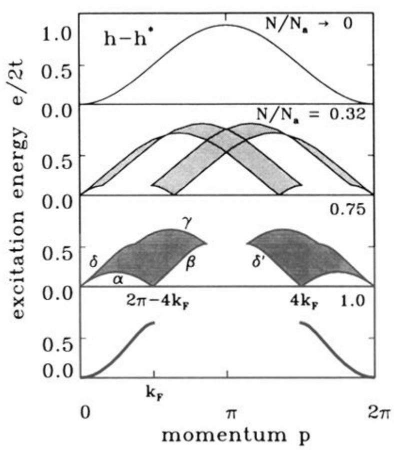

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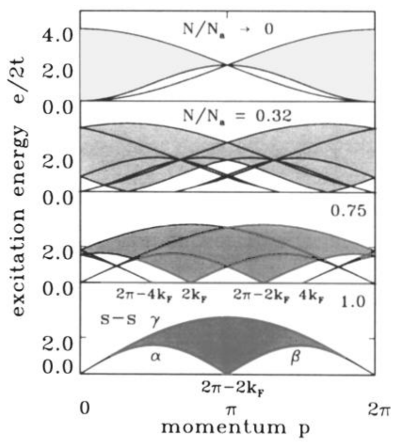

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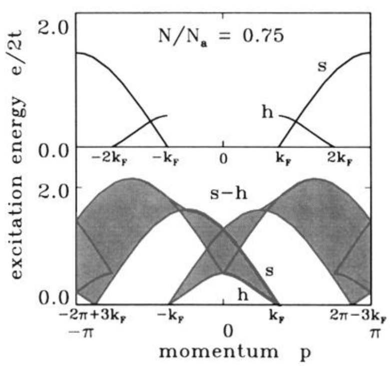

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