Nonlinear Dynamics of a Forced Thermoacoustic Oscillation

T. Yazaki, S. Sugioka, F. Mizutani,[a] and H. Mamada
Department of Physics, Aichi University of Education, Igaya-cho, Kariya-shi Aichi-ken, 448, Japan
(Received 27 December 1989)

A spontaneous acoustic oscillation of a gas column induced by temperature gradients is periodically perturbed by an external force. Nonlinear coupling between two oscillating modes can generate abundant nonlinear phenomena, quasiperiodicity, frequency locking, onset of chaos, and quenching phenomenon. Adjusting the frequency ratio to the golden mean, we study the scaling universality for the quasiperiodic transition to chaos. The first experiment in acoustics demonstrates that thermoacoustic systems follow universal scaling properties of the circle map in spite of the complexity of the system.

PACS numbers: 47.25.Ae, 05.45.+b, 43.90.+v

Universal scaling properties for the quasiperiodic transition to chaos have been predicted by the circle map.[1-3] No matter how complex physical systems are or even if basic equations characterizing nonlinear systems are not known, the behaviors of the system at the onset of chaos may be able to be effectively described by scaling universalities predicted by the map. The relevance of predictions has been tested and supported in a few experimental systems, Rayleigh-Bénard convection,[4] and solid-state physics.[5,6] There exists, however, an experimental system with distinct deviation from the universalities predicted by the circle map.[7]

The experimental system attracting our interest is a hydrodynamic one called "Taconis oscillation"[8] in thermoacoustics. When a gas column confined in a long tube is subjected to strong temperature gradients, it spontaneously oscillates with considerable amplitudes ($\sim 10^4$ Pa or more). Theoretical studies, especially stability analysis, have been carried out by Rott.[9] We would like to determine whether the scaling universalities predicted by the map are applicable to such complex thermoacoustics. In a previous Letter,[10] we reported that two or three different oscillatory modes of a gas column can be simultaneously induced in a tube and competition between them leads to quasiperiodic and chaotic oscillations. By substituting the external excitation for one of two competing modes, we can investigate the universal properties for the quasiperiodic transition to chaos at any desired winding number. In this Letter, we will present some experimental evidence in favor of the applicability of the map to thermoacoustics under a particular value of the winding number, the golden mean $\sigma=(\sqrt{5}-1)/2$. Still more, which variable of the map corresponds to the observable of experiments, hydrodynamic variable, the angle or its projection? We will also give some explanation for this question argued by Fein, Heutmaker, and Gollub[4] by examining the universal form of the power spectrum. This Letter is the first experimental report on scaling universalities for the quasiperiodic transition to chaos in acoustic systems of a fluid.

The experimental arrangement was shown in previous papers.[8,10] A gas is contained in a long stainless-steel tube (whole length 2.8 m, inner radius $r=1.2$ mm) with a symmetrical steplike temperature distribution along its axis. Both ends which are at room temperature T_H are closed by a small pressure transducer and a stainless-steel dynamic bellows, respectively. The cold part, smoothly bent as U shaped near the midpoint of the tube, is immersed into liquid helium ($T_C=4.2$ K). The working gas is helium. The ratio of the tube lengths of the cold part to the warm, ξ, plays an important role to limit the number of modes thermally induced; two and three different modes are simultaneously excited for $\xi=0.5$ and 0.3, respectively, because of the intersection of their stability curves. For ξ larger than 1, higher modes are hard to excite and frozen out, and only the fundamental is singled out as an excited mode. Thus the value of ξ is 1 in our experimental design. A woofer speaker (to which an ac voltage with frequency f_e through a power amplifier from a synthesizer is applied) is attached to the bellows. This can produce a simple harmonic oscillation of a gas column. Thus the thermally driven oscillation can be dynamically coupled with the one mechanically driven under the desired winding number. We could tune the winding number to the golden mean within the accuracy of 10^{-4}. The signal voltage was digitized by 12-bit analog-to-digital converters. Two types of time series of 16 384 points sampled by a suitable time and an external driving frequency were analyzed by a computer in order for us to obtain detailed power spectra and strobed attractors. The phase-locking states were ascertained by direct observations of the Poincaré section in real time. We could easily search the $\frac{13}{21}$ locking state on a display unit.

The amplitude of Taconis oscillation is controlled by two dimensionless parameters,[9] the temperature ratio T_H/T_C, and $R=r\sqrt{a_c/\alpha_c l_c}$, where a_c and α_c are adiabatic sound velocity and thermal diffusivity of the gas at the cold part, respectively, and l_c is a half tube length of the cold part. Keeping the temperature ratio constant (≈ 70.4), we increase R by a gradual increase of the density of the gas. After R goes beyond the critical value R_c ($=14.4$) a gas column spontaneously begins to oscillate with the frequency f_0. We chose $R=17.2$, where the oscillation has a high enough amplitude for us to observe the onset of chaos. In Fig. 1(a) we show the spectrum of Taconis oscillation with no external force. Adjusting the "dressed winding number" to a constant, the golden mean, and keeping the control parameter R fixed, we gradually increase the external amplitude. In Figs. 1(b) and 1(c), we show power spectra well below and just at the critical point, the nearest point to the onset of chaos. With the approach to the critical point, the spectrum shows an increase of visible noise floor in addition to the increase of higher-order mixing components. The determination of the transition to chaos was also confirmed by the drastic change of the Poincaré section. Figure 2 shows strong subcritical and just-critical Poincaré sections corresponding to Figs. 1(b) and 1(c), which are reconstructed from time series $V(t)$ strobed with the driving frequency and are nonintersecting in embedding three dimensions. The Poincaré section is a well-defined closed loop with no fold for the subcritical, and just above the critical point wrinkles start appearing on the section and the invariant two torus is just broken down.

In general, nonlinear systems tend to display more complicated behavior as they undergo harder external perturbation. Our experimental system, however, never exhibits this tendency. After the onset of chaos is achieved, further increases beyond a value of the external amplitude never lead to a further increase of the number of combination peaks, but decrease of the number of them in spite of enough large amplitude; and still more, the excitation tends to dominate a flow in the system. The external amplitude has some band dependent of R for observation of chaos. For smaller R where the amplitude of the instability is not so large, quasiperiodicity and frequency locking are observed, but the number of the linear combination peaks is small and chaos can be never observed. As the external amplitude goes over a critical value, Taconis oscillation is quenched by the external force. Then the spectral peak of the instability begins to fall in and at last vanishes. This is shown in Fig. 1(d). The excitation also starts to dominate a flow completely in the system. It bears a striking resemblance to the phenomenon reported in a forced Rayleigh-Bénard convection. Stavans[11] has predicted that a multicritical point exists in a generalized parameter space with the amplitudes and the frequencies of two oscillators and the stable critical points for the onset of chaos can be observed only in the large-amplitude region. In order for the quasiperiodic transition to chaos to be observed, the selection of the instability amplitude larger than a critical amplitude dependent of the winding number is required in addition to the limitation of the external amplitude. Quenching phenomenon has been indicated in the self-excited system to which a periodical external force is applied; for example, a forced van der Pol equation.

The scaling universality predicted by the circle map can be seen in the power spectrum at the critical point.[1] The spectrum for the rotation angle of the map has a self-similar structure and its envelope is scaled by the power law f^2 when the winding number is fixed at the golden mean. Previous experiments[4,5] have supported this scaling power law. To confirm this, we reconstruct the critical power spectrum shown in Fig. 1(c). The result is represented in Fig. 3(b), where the power divided by f^2 is plotted on logarithmic scale for the frequency axis. The result demonstrates the self-similar structure and the scaling with f^2 in the low-frequency region. At least the envelope of the spectral peaks for the first generation (labeled 1), which is all linear combination of f_0

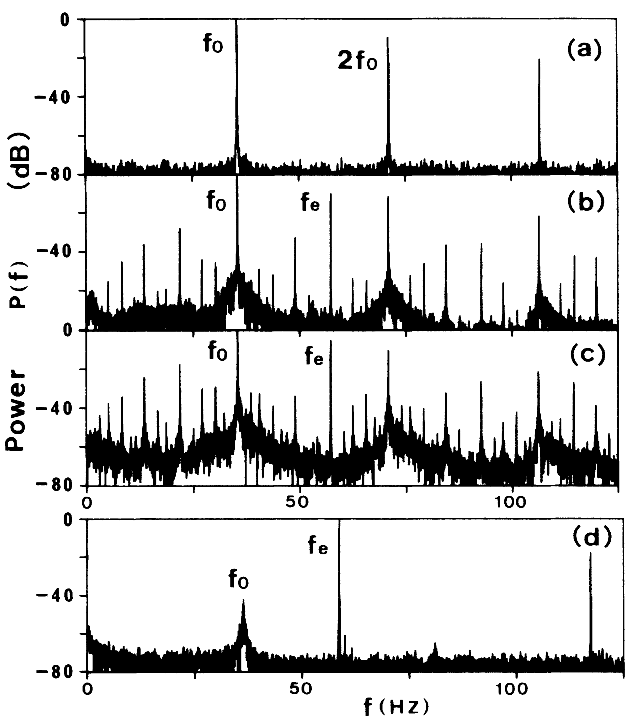

FIG. 1. Power spectra of pressure fluctuations: (a) Taconis oscillation with no external force, (b) quasiperiodic, (c) onset of chaos, and (d) quenching phenomenon.

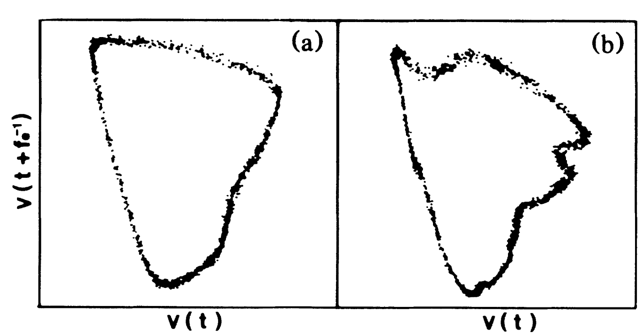

FIG. 2. Experimental Poincaré sections projected onto a plane: (a) subcritical and (b) critical corresponding to Figs. 1(b) and 1(c), respectively.

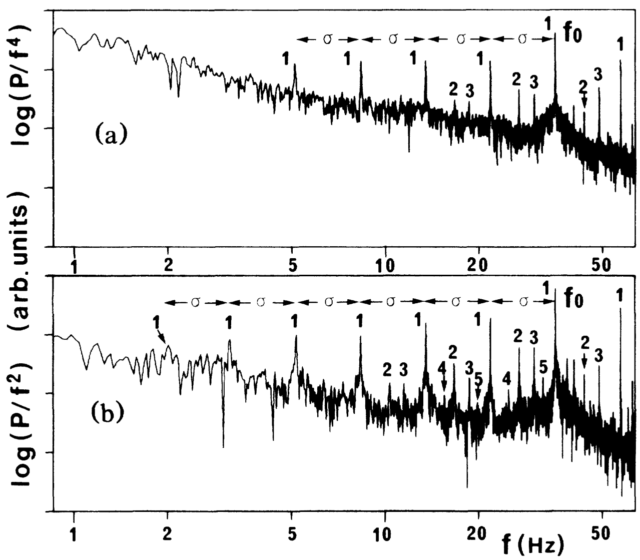

FIG. 3. Scaled power spectra: (a) subcritical and (b) critical corresponding to Figs. 1(b) and 1(c), respectively.

and f_e with successive coefficients of the Fibonacci sequence for the seed (1,1), is found to be approximately constant in height. Peaks of the first generation can be recognized up to σ^7 in spite of no averaging. Other spectral peaks (labeled by 2,3,...) generated from Fibonacci sequence for different seeds do not seem to have the same height. This may be attributed to a slight deviation from the criticality. The experiment seems to suggest that the observable, hydrodynamic variable, corresponds to the angle of the map rather than the projection along an axis in the phase space, whose critical power spectrum varies as f^4 scaling power law in the envelope. Fein, Heutmaker, and Gollub[4] and Mori,[12] however, indicate that the observable of experiments should correspond to the projection of the angle rather than the angle itself. This problem may be solved by the analysis of the power spectrum below the critical point, where the envelope follows the scaling with a different power law. The scaling of the power spectrum for the subcritical has not been reported in previous experiments. The power spectrum in Fig. 1(b) scaled by f^4 is shown in Fig. 3(a). The spectrum for the subcritical follows the scaling with f^4, at least for the first generation, instead of that with f^2 for the criticality. As the system approaches the onset of chaos, the scaling power law changes from f^4 to f^2. Such a change should be attributed to the drastic change of the Poincaré section. Antoranz and Mori[13] have theoretically studied the power spectrum of the Cartesian coordinate of the intersection point on the Poincaré section, using a linear map for the rotation angle and the simplified model for the radius of the point characterizing wrinkles on a torus. They have predicted that the spectral peaks at low frequency follow the scaling with the power law f^2 or f^4 according to whether the Poincaré section has sharp or mild wrinkles. The change of the scaling power law observed can be explained by their model. Thus our results demonstrate that the observable of experiments does correspond to the projection along an axis in the phase space.

Phase-locked regions ordered in the ratio of the adjacent number of the Fibonacci sequence with the seed (1,1) were drawn up according to Farey composition (not shown here). They were a typical diagram known as "Arnol'd tongues" predicted by the circle map. We focused our attention on the locking structure, the widths of tongues along the critical line. Sweeping the driving frequency around the golden mean under constant external amplitude and monitoring the Poincaré section, we measured the distance S (55.79–57.77 Hz) between two locked-band parents (3:5 and 5:8) and the distance S_1 (55.79–56.60 Hz) and S_2 (56.76–57.77 Hz) which are interval lengths between the daughter band (8:13) and two parents, respectively. The tongues are predicted to form a Cantor set along the critical line. The fractal dimension D of the set is *estimated* from the equation $(S_1/S)^D+(S_2/S)^D\approx 1$. We obtained $D\approx 0.89$ with an experimental error of ± 0.02, which is close to the theoretical value[2] of 0.868.

A global scaling property of the attractor is described by the multifractal spectrum[3] $f(\alpha)$, which is a fractal dimension of the set of the singularity with scaling index α of the measure. In order to efficiently calculate $f(\alpha)$ from short data sets we used an indirect method, recurrence-time approximation, proposed by Jensen *et al.*[4] The probability $P_i(l)$ that other points fall within a small distance l of a given point i on the attractor was estimated from the inverse of the averaged recurrence time $m_i(l)$. According to whether the concentration of the point is high or low, the recurrence time is short or long. We determined $f(\alpha)$ using the relation $\langle m_i(l)^{1-q}\rangle \sim l^{\tau}$, for small distance l, where the angular brackets show the average over all points ($\sim$3000 points) on the attractor (Fig. 2). For many values of q, the values of $\tau(q)$ were determined by fitting the log-log plot of $\langle m_i(l)^{1-q}\rangle$ *vs* l to a straight line. The $f(\alpha)$ spectrum was calculated from $\tau(q)$ through the Legendre transformation.[3] The results are shown in Fig. 4, where solid and open circles are for the critical and subcritical attractors in Fig. 2, respectively. Large experimental error bars at the right-hand branch corresponding to rarefied regions are attributed to the variation of the fitting range of l. The solid curve is the theoretical $f(\alpha)$ spectrum by the sine circle map at the critical point. The experiment supports that the Taconis oscillation belongs to the same universality class as the circle map in spite of the complexity of thermoacoustics. A small deviation from the criticality takes place at the drastic change of $f(\alpha)$. $f(\alpha)$ for the subcritical shown in Fig. 4 corresponds to that for $\epsilon\approx 10^{-3}$ deviation from the criticality according to the theory of Arneodo and Holschneider.[14]

Thermoacoustic phenomena[8] have been studied for

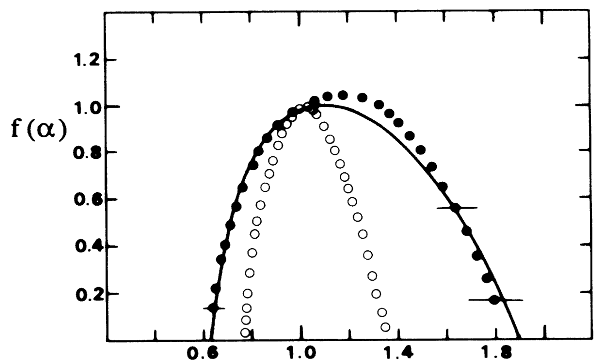

FIG. 4. Experimental $f(\alpha)$ spectra for the subcritical (○) and critical (●). The solid curve is the theoretical $f(\alpha)$ predicted by the sine circle map with the golden-mean winding number at the onset of chaos.

more than two centuries. The progress of the investigation was not speedy because of great complexity and for lack of systematic experimental work. Recently, there has been a rise in scientific activity related to the concept of thermoacoustics. Oscillations by heat have been of great interest in many physical systems, astronomy, fields of application, and so on.[15] The applicability of the map universality to the system will enable us to effectively predict the qualitative behavior of nonlinear thermoacoustic phenomena even if basic equations are not known or difficult to be solved. This is very important for the application of thermoacoustics.

This work was performed at the Instrument and the Cryogenic Centers, the Institute for Molecular Science. We would like to thank K. Hayasaka and K. Kato for the preparation of liquid helium.

[(a)]Present address: National Institute for Physiological Science, Okazaki, 444, Japan.

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