# Generation of Fock States of the Electromagnetic Field in a High-$Q$ Cavity through the Anderson Localization

J. R. Kuklinski
*Institute for Theoretical Physics, Polish Academy of Science, 02-668 Warsaw, Aleja Lotnikow 32/46, Warsaw, Poland*
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We investigate a high-$Q$ cavity partially filled by a medium with $\chi^3$ nonlinear susceptibility with a coherent signal having the form of a series of short pulses injected into it. We show that the electromagnetic field which builds up in the cavity has strong nonclassical features. For appropriate parameters Anderson localization occurs leading to the generation of states with a narrow photon-number probability distribution. We demonstrate that by adiabatic changing of the size of the cavity one may generate from the vacuum a nearly perfect Fock state of a chosen mode.

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Investigations concerning generation of nonclassical states of the electromagnetic field are still attracting a lot of attention. Up to now various methods of generation of squeezed and antibunched light have been proposed and tested in experiment.[1,2] However, the generation of the electromagnetic field in a pure Fock state remains a particularly interesting goal.[3–6] This subject has been widely studied, namely, in the context of recent investigations of the single-atom micromaser consisting of a very-high-$Q$ microwave cavity excited by a beam of inverted atoms.[3,4] Nowadays the lifetime of such a microwave cavity can attain the order of seconds[4]—in that case the cavity field excited by means of single-emission acts of atoms passing through the cavity can be close to a Fock state.[3,4] Another method of generation of Fock states of the electromagnetic field making use of an interferometer with an optical Kerr medium (nonlinear Mach-Zendher interferometer) was proposed by Kitagawa and Yamamoto.[5]

In this paper we are investigating the possibility of generation of a nearly perfect Fock state of the electromagnetic field in a high-$Q$ cavity partially filled by a medium with $\chi^3$ nonlinear susceptibility with a series of coherent pulses injected into it. We assume that the active medium is not resonant to the excited cavity mode and that there is no interaction between various cavity modes induced by the optical nonlinearity (we describe our system with the help of an effective Hamiltonian involving only the electromagnetic field). In such conditions the cavity with an injected pulsed signal is found to be similar to a kicked rotator with nonlinearity.[7] Like in the latter system we found a strong localization when states with a very narrow distribution of the photon-number probability are generated. Finally, we consider the case when the size of the cavity is adiabatically varied—for such a case one can generate a nearly perfect Fock state in a chosen mode of the electromagnetic field in the cavity.[8]

Consider a high-$Q$ cavity partially filled by an active medium with the third-order nonlinear susceptibility. A microwave cavity similar to the one used in recent Rydberg atoms experiments[4,9,10] or maybe a high-finess optical resonator achieved in the near future could be a physical system of interest. We assume that a coherent signal having the form of a periodic series of identical short pulses with central frequencies $\omega_L$ and the pulse duration $\tau$ is injected into the cavity. The driving frequency $\omega_L$ is taken to be resonant with a chosen cavity mode. Thus the injected signal will excite the resonant mode and possibly also neighboring modes.[11] Our next assumption is that there is no interaction between various cavity modes induced by the interaction of light with the optically active medium, i.e., that excitations are not transferred from one mode to another. The simplest situation of this type occurs when only the lowest cavity mode is excited and there is no nonlinear resonances of the type $\omega_{n_1}+\omega_{n_2}=2\omega_{n_3}$ or $\omega_{n_1}+\omega_{n_2}+\omega_{n_3}=\omega_{n_4}$ with higher modes. Another case of interest is when many neighboring modes are excited but the nonlinear transfer between these and other modes is blocked because of lack of (nonlinear) resonance. We should stress that the model situation, when an active medium with the third-order nonlinear susceptibility interacts with the electromagnetic field without introducing a transfer of excitations from one mode to another, has been discussed several times in the literature,[12] namely, also in context of the generation of nonclassical states of light,[13,14] but necessary conditions to avoid the mode-mode interaction have not been investigated in detail.

A separate problem is the consideration of the dynamic of the electrodynamic field in a cavity with a volume changing in time. The general problem when the size of the cavity changes arbitrarily is very complicated. However, in the situation of interest the size of the cavity is varied very slowly (for simplicity, we assume that the cavity volume is not changed when a pulse is injected into the cavity). For each time instant we have a different boundary condition and therefore a different set of eigenmodes, i.e., a different set of eigenvectors of the (many-particle) Hamiltonian. Assuming that the relative rate of change of the cavity size is very small in comparison to the energy spacing of the system, i.e., that

$$\left|\frac{d\hat{H}(\lambda)}{d\lambda}\hat{H}^{-1}(\lambda)\frac{d\lambda}{dt}\right| = \left|\frac{d\omega_n(\lambda)}{d\lambda}\omega_n^{-1}(\lambda)\frac{d\lambda}{dt}\right| = \frac{dV}{dt}V^{-1} \ll \Delta\omega ,$$

where $\lambda = V$ is the quantization volume, $h\omega_n(\lambda)$ are the eigenvalues of $\hat{H}(\lambda)$, and $h\Delta\omega$ is the energy spacing, we can invoke the quantum adiabatic theorem[15] and approximate the time-dependent wave vector of our system by the following adiabatic solution. In other words, if the system is initially in any of its eigenstates $|\Psi(t_0)\rangle = |\Phi_n(\lambda(t_0))\rangle$ (i.e., in a Fock state), it will remain in a time-dependent eigenstate all throughout its evolution, i.e.,

$$|\Psi(t)\rangle \cong |\Phi_n(\lambda(t))\rangle \exp\left[-i\int_{t_0}^{t}\omega_n(\lambda(t'))dt'\right] .$$

Following this simple argument, we will describe our system in the limit $dV(V\Delta\omega)^{-1}/dt \ll 1$ with the help of a time-dependent Hamiltonian (together with a set of time-dependent eigenfunctions) which is at any time instant identical to the one found for a fixed volume of the cavity. We stress that an important point for consideration of such an adiabatic (or quasistatic) approximation is that we have assumed no nonlinear resonances allowing a conversion of excitations from the excited mode of our interest to other modes.

Following the previous discussion we will investigate the dynamics of a single mode of the electromagnetic field. The Liouville equation for our system reads ($\rho$ denotes the density matrix of the single mode of the electromagnetic field, the Hamiltonian is presented under the adiabatic approximation versus the changes of the cavity volume, and in the rotating-wave approximation with respect to the optical frequency, therefore, rapidly oscillating terms are not present):

$$d_t\rho = i[\rho, H_0 + H_1] + L_\Gamma\rho , \tag{1}$$

$$H_0 = \Delta(t)a^\dagger a + \chi a^\dagger a^\dagger aa , \tag{2a}$$

$$H_1 = \Omega(t)(a^\dagger + a) , \tag{2b}$$

$$L_\Gamma\rho = 2\Gamma(a^\dagger\rho a - \rho a^\dagger a - a^\dagger a\rho) . \tag{2c}$$

As we see parameters $\Delta(t)$ and $\chi$ are describing the time-dependent detuning of the cavity from the driving frequency and the nonlinearity strength, respectively. The term proportional to $\Gamma$ describes the absorption of light by cavity mirrors in the limit of zero temperature (see, for example, Ref. 16) while $H_1$ describes the coherent electromagnetic wave injected into the cavity. The injected coherent signal has the form of a periodic series of pulses with the pulse duration $\tau$ and the periodicity $T$. Assuming that the pulse duration is shorter than any relevant time scale and pulses are strong enough, i.e., that $\tau/\Omega_0 \ll \Gamma^{-1}, \chi^{-1}, \Delta^{-1}, T$, one may approximate[17]

$$\Omega(t) \cong \Omega_0 \sum_{n=0}^{\infty} \delta(t - nT) .$$

Our analysis is based on numerical solutions of the Liouville equation (we assume $T=1$, e.g., we scale all time-dependent quantities in unit of $T$). Since we investigate the regime of a small cavity damping, a Trotter-like approximation is performed when solving Eq. (1) numerically. The evolution of the density matrix over the time interval $T$ is found by using a Trotter approximation on the single time-step propagation operator, e.g., we take $\exp[(L_\Gamma + L_I)T] \cong \exp(L_\Gamma T)\exp(L_I T)$ ($L_I$ denotes the Liouville operator describing interaction without the cavity damping; see also Ref. 18). It can be immediately seen that when the nonlinearity vanishes ($\chi = 0$) a coherent state builds up in the cavity (a steady-state limit is reached on a time scale $T_c = 1/\Gamma$, with a mean photon number approximately equal to $N_{ph} = \Omega_0/\Gamma$). The situation is different when the nonlinear medium is present ($\chi \neq 0$)—the field in the cavity may saturate at an intensity much lower than previously ($N_{ph} \ll \Omega_0/\Gamma$) and it is no longer a classical-like coherent state. This effect may be viewed as a nonlinear saturation. In order to get a better understanding of it, first, we study briefly the model case when the cavity field is initially in a pure Fock state. In such a case the interaction can bring us to a state with an exponentially localized probability distribution as presented in Fig. 1—a similar effect is well known from the quantum behavior of a kicked rotator.[7] We can see that the localization length will increase when the kicks strength $\Omega_0$ is increased (for $\Omega_0$ large enough the localization disappears); large cavity losses are also supposed to destroy the localized state. In Fig. 2 we present the dependence of the steady-state photon-number distribution spread $\sigma_n$

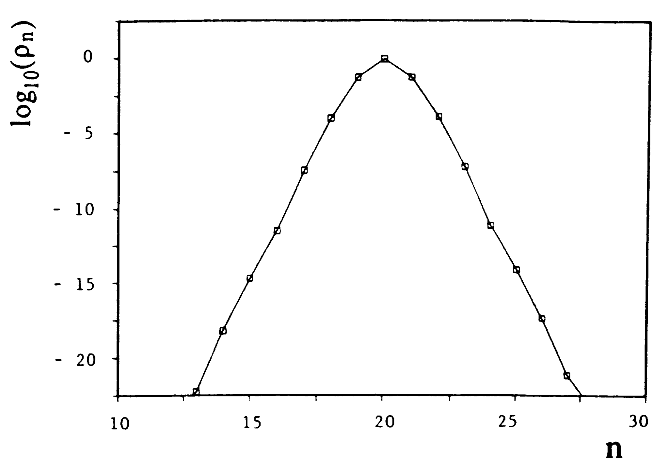


FIG. 1. The photon-number probability distribution $\rho_n$ vs $n$ on a logarithmic scale. The interaction time is $T_{int} = 10^3 T$; other parameters are $\Omega_0 = 10^{-2}$, $\chi = 0.32T^{-1}$, $\Delta = 10/0.32$, and $\Gamma = 10^{-6}T^{-1}$ (i.e., $Q = 10^6 \nu T$). The initial state is a Fock state with $n_0 = 20$.

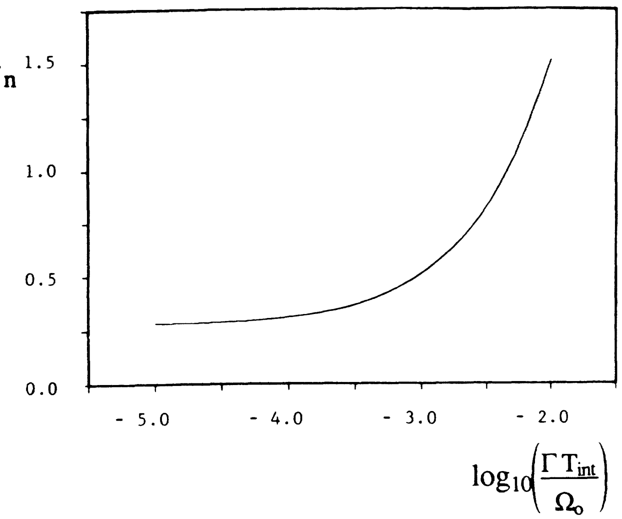


FIG. 2. The variance of the photon-number distribution $\sigma_n^2=\langle(n-\langle n\rangle)^2\rangle$ vs $\Gamma T/\Omega_0$. The parameters are $\Omega_0=10^{-2}$, $\chi=0.32T^{-1}$, $\Delta=5/0.32$, and $T_{\rm int}=10^3T$. The initial state is a Fock state with $n_0=10$.

on the cavity damping rate $\Gamma$—the effect of cavity losses is visualized when $\Gamma$ exceeds a certain critical order of magnitude. For smaller damping rates our system behaves exactly as in the lossless case—for much larger losses the localization effect is destroyed.[19] We stress also that the localization becomes more sensitive on $\Gamma$ when the kicks strength is decreased; i.e., for a given cavity loss rate we cannot obtain the arbitrarily strong localization.

Next, we consider the case when initially the field in the cavity is in the vacuum state. We switch on the interaction and change adiabatically the cavity detuning $\Delta(t)=\theta(t-t_0)\Delta_0(t-t_0)/(t_1-t_0)$ (i.e., we slowly change the cavity size). In such conditions a localized state is generated around the vacuum state (we have taken $\chi$ and $\Omega_0$ the same as in Fig. 1)—next this localized state is adiabatically moved to an arbitrarily excited state because of changing $\Delta(t)$. In such a way we can get a nearly perfect Fock state starting from the vacuum state; the numerical results showing such behavior are presented in Fig. 3.

At this point it is interesting to compare our calculations to recent Rydberg-atom-maser experiments.[4,9,10] Let us consider a cavity with $Q=10^{10}$, the lowest cavity eigenfrequency $\nu\cong 10^{10}\ {\rm s}^{-1}$ and a series of $4\times10^4$ short coherent pulses with the pulse duration $\tau=25\nu^{-1}$ and the pulse interval $T=10\tau$, which is injected resonantly to the lowest cavity mode ($\Gamma=\nu/Q$, we stress that in the recent microwave cavity experimental achievement a factor $Q=3\times10^{10}$ was attained and the mode frequency was $\nu\cong2\times10^{10}\ {\rm s}^{-1}$). It can be easily verified by inspection that these parameters correspond to the numerical calculations presented in Fig. 3, providing that the approximation of the driving field by $\delta$ functions is applied.[20] As we see the regime of parameters needed to observe Anderson localization seems to be not too far from the recent achievement in Rydberg-atom-maser experiments.[4,9,10] However, an important point is the choice of an active optical medium, which will not induce relatively high absorption of light. It seems that a bunch of Rydberg atoms quasiresonant to the excited mode will be a good candidate. The spontaneous emission of atoms

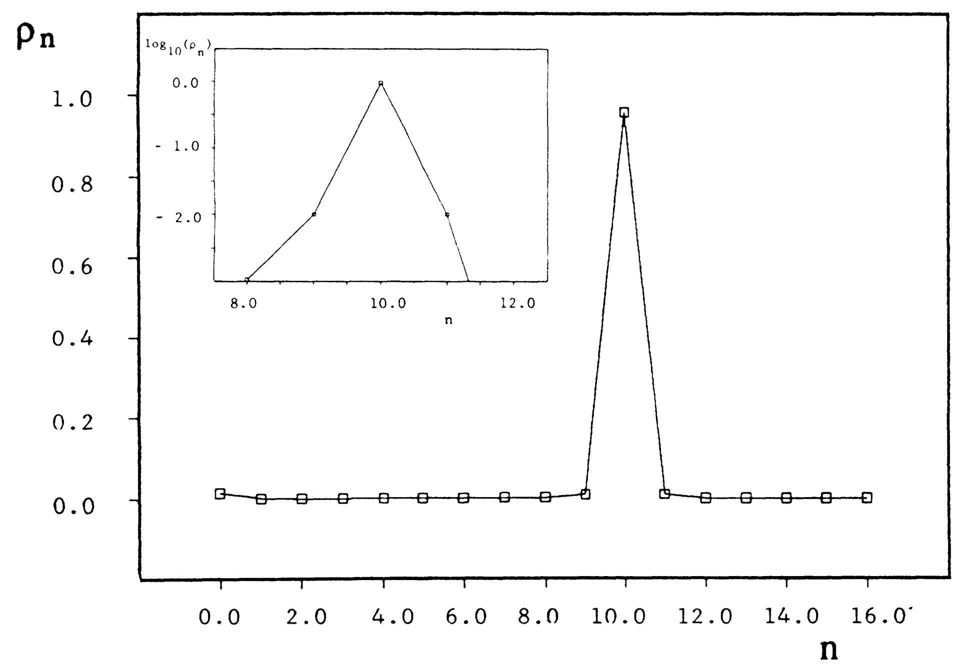


FIG. 3. The photon-number probability distribution $\rho_n$ vs $n$. The initial state is vacuum and the cavity size is changed during the interaction ($t_0=0$, $t_1=4\times10^4T$, $\Delta_0=5/0.32$). The other parameters are $\Omega_0=10^{-2}T^{-1}$, $\chi=0.32T^{-1}$, and $\Gamma=2.5\times10^{-8}T^{-1}$ (for $\nu=2.5\times10^2T^{-1}$ we get $Q=10^{10}$). Inset: A logarithmic plot is presented.

is in such a case "blocked" by the cavity and the remaining potential mechanisms of dissipation (assuming that there are no nonlinear resonances allowing the transfer of excitations from the excited mode to unexcited ones) are collision-assisted light-atom interactions. The evaluation of the number of atoms and their detuning from the cavity field, necessary to provide sufficient $\chi^3$ susceptibility together with a low collision rate, requires the consideration of various effects (namely, of dipole-dipole interaction of randomly distributed atoms) and will be presented in a future publication.[21]

Since the strong localization takes place when cavity losses are nearly negligible, we can give some simple intuitive arguments about the localization effect in the frame of the Schrödinger equation describing a lossless cavity [Eq. (3)]. We have assumed the injected field to be a sum of $\delta$ functions; therefore we get an evolution operator in a form characteristic for kicked systems [Eq. (4)]

$$
\begin{aligned}
&|\Psi(t)\rangle=\sum_{j=0}^{\infty}\phi_j(t)\,|j\rangle\,,\\
&d_t\phi_j(t)=-iE_j\phi_j(t)\\
&\qquad\qquad+i\Omega(t)[\sqrt{j}\,\phi_{j-1}(t)+\sqrt{j+1}\,\phi_{j+1}(t)]\,,\\
&E_j=j\Delta(t)+j(j-1)\chi\,,
\end{aligned}
\tag{3}
$$

$$
|\Psi(t_n=nT)\rangle=\exp(-iH_0T)\exp(-iH_1T)\,|\Psi(t_{n-1})\rangle\,.
\tag{4}
$$

Note that the dynamics of our system will remain unchanged if we replace the "energy" $E_j$ by $\tilde{E}_j=\mathrm{mod}(E_j,2\pi/T)$. We immediately see that $\tilde{E}_j$ can be somehow randomly distributed on the interval $(0,2\pi/T)$. The necessary condition for such random behavior is that $T/2\pi$ and $\chi$ are incommensurate. In this case the Anderson localization[22] is expected to occur similarly to the well-known example of the kicked rotator (see, for example, Ref. 7, and references therein).

To summarize, we have shown that a strong localization effect occurs when a series of short coherent pulses is injected into a high-$Q$ cavity partially filled by an active nonlinear medium. Further, the localization effect can be used for generation of Fock states of the electromagnetic field from an initial vacuum state when the cavity size is slowly changed while consecutive pulses are injected. The necessary condition to observe described effects is that the cavity lifetime should be very large and that there are no nonlinear resonances between various modes. The analysis presented in this paper was performed under many simplifying approximations (namely, of very short pulses) and the source of randomness leading to the localization effect was the incommensuration of two physical parameters. A more elaborate numerical analysis, performed without the $\delta$ approximation and including the case when random fluctuations of pulse strength and phase are responsible for Anderson localization, will be presented in a future publication[21] (a more detailed study of the situation described here will be presented).

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[11] We immediately see that a single mode will be excited when the pulse duration length $\tau$ will be shorter than the cavity round-trip time $T_r$. In the opposite case the number of excited modes is roughly equal to $N_{\rm mode}\approx\tau/T_r$.

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[17] Assuming that our injected signal has the form of square pulses with the pulse duration $\tau$ and the maximum strength $\Omega=\Omega_0\tau$, the approximation of interaction by $\delta$ functions is valid when $\Omega$ is strong enough to dominate the interaction Hamiltonian when the pulse is switched on, i.e., when $\Omega\gg\Gamma,\chi,\Delta$.

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[19] The quasi-steady-state corresponding to larger damping rates in Fig. 2 still has a narrow photon-number distribution, but this distribution deviates from the exponential shape observed in Fig. 1.

[20] We also find out that if the frequency separation from the nearest nonlinear resonance is $\Delta\omega_R\gtrsim10^{-4}\omega_0$ $[\omega_0=2\pi\nu$, $\Delta\omega=\min(\Delta\omega_R,\omega_0)]$, the adiabatic approximation can be applied to the electromagnetic field in the cavity with changing volume since $dV(V\Delta\omega)^{-1}/dt\lesssim2\times10^{-4}$ $(dVV^{-1}/dt=T_{\rm int}^{-1})$. We assume that the volume of the cavity is changing with a constant rate.

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