Superdiffusion in Random Velocity Fields

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We discuss the *superdiffuse* motion of a random walk in a medium containing random velocity fields. For a two-dimensional layered medium with y-dependent random velocities in the x direction $u(y)$, $\langle x^2(t)\rangle \sim t^{2\nu}$, with $2\nu = \frac{3}{2}$, and with strong sample-to-sample fluctuations. The probability distribution of displacements, averaged over environments, takes a non-Gaussian scaling form at large time, $\langle P(x,t)\rangle \sim t^{-3/4} f(x/t^{3/4})$, where $f(u) \sim \exp(-u^\delta)$ for $u \gg 1$, with $\delta = \frac{4}{3}$. For an isotropic two-dimensional medium with $u_x(y) = f(y)$ and $u_y(x) = f(x)$, we find $\nu = \frac{2}{3}$ and $\delta = (1-\nu)^{-1} = 3$.

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Stochastic transport in random media is often *subdiffusive*, where the mean-square displacement $\langle x^2(t)\rangle$ grows more slowly than linearly with time (see, e.g., Refs. 1-3 for recent reviews). In this Letter, we discuss a simple and general mechanism, based on the coupling between diffusion and convection by spatially random, but temporally static velocity fields, that leads to the complementary situation of *superdiffusion,* where $\langle x^2(t)\rangle$ grows *faster* than linearly in time.

Superdiffusive transport has been treated previously in a variety of contexts, such as chaotic systems,[4] turbulence,[5] flow in fractal geometries,[6] and Levy flights.[7] In the latter case, both subdiffusion and superdiffusion can arise by choosing a sufficiently singular single-step probability distribution. Here, we discuss a situation, relevant to ground water transport in geological aquifers,[8] where superdiffusion arises naturally, rather than being built into microscopic-level transport laws. Measurements of tracer dispersion in aquifers indicate that the dispersivity systematically increases with distance between source and sink.[8] A theoretical model involving random velocity fields was constructed by Matheron and de Marsily[9] that yields a superdiffusive spread of a tracer pulse. We extend this treatment in several important respects to gain a deeper insight into the role that random velocity fields play in superdiffusion.

The basic phenomenon can be appreciated by considering a two-dimensional stratified porous medium consisting of distinct parallel layers with different transport properties in each layer [Fig. 1(a)]. When a pressure drop is applied along the strata, the longitudinal fluid velocity correspondingly varies from layer to layer. In a center-of-mass frame of reference, therefore, the steady velocities in the x direction are random zero mean functions of the transverse coordinate y. Although the longitudinal bias averaged over an infinite number of layers is zero, the average over the finite number of layers that a random walk visits is a fluctuating quantity which is a decreasing function of the number of layers sampled. This nonvanishing bias underlies superdiffusive transport.[9-12]

Within a continuum description, the random-walk motion is accounted for by the Langevin equations,

$$dx/dt = u(y(t)), \quad dy/dt = \eta(t)\,, \qquad (1)$$

in which a random walker undergoes pure diffusion along y [$\langle \eta(t)\eta(t')\rangle = 2D\delta(t-t')$, where D is the transverse diffusion coefficient] and is passively carried by the quenched random convection field $u(y)$ along x. Diffusion noise in the x direction is subdominant and thus neglected with respect to the random convection. For simplicity, we take the convection field to be a Gaussian white noise in space, $\langle u_x(y)u_x(y')\rangle_c = \sigma\delta(y-y')$, where $\langle \cdots \rangle_c$ denotes an average over all velocity configurations of the medium.

To compute the moments of the longitudinal displacement, note that for a given walk in a fixed environment,

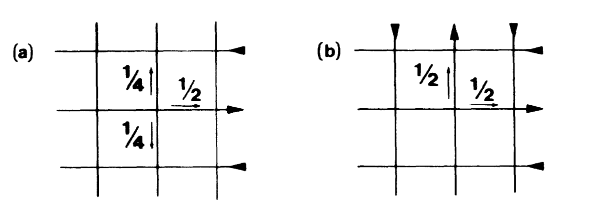

FIG. 1. (a) The random stratified medium and (b) the random isotropic medium on the square lattice. The hopping rules at a typical lattice site are indicated. In (a), the layers consist of contiguous rows of the same orientation.

the longitudinal position at time t can be written formally as

$$x(t)=\int_0^t u(y(t'))dt'=\int_{-\infty}^{+\infty}\mathcal{N}(y,t)u(y)dy\,, \qquad (2)$$

where $\mathcal{N}(y,t)=\int_0^t dt'\delta(y-y(t'))$ is the number of times that the transverse Brownian motion $y(t)$ visits layer y after time t, having started at $y=0$. The superdiffusive behavior of this model can be understood by the following heuristic estimate of $x_{\rm rms}(t)$ from Eq. (2). The quantity $\mathcal{N}(y,t)$ is of the order of the elapsed time t divided by the number of y layers encountered, $\sqrt{2Dt}$. Further, the rms value of $\int dy\, u(y)$ is roughly $\sigma^{1/2}\times(2Dt)^{1/4}$, where the latter factor is the square root of the typical range of the integral. Thus

$$x_{\rm rms}(t)\sim\langle u_x\rangle_t t\sim\sigma^{1/2}D^{-1/4}t^{3/4}\,, \qquad (3)$$

a remarkable result which was apparently first derived in Ref. 9 and which also can be derived in terms of a power-law decay of the longitudinal velocity correlation function.[10]

To justify and extend this intuitive argument to higher moments of the longitudinal displacement, we write these moments as

$$\langle\langle x^n(t)\rangle_w\rangle_c=n!\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\langle\langle u(y(t_1))\cdots u(y(t_n))\rangle_w\rangle_c\,. \qquad (4)$$

The double angular brackets indicate that one should first average over all transverse Brownian trajectories for a given configuration of random velocities, and then average over all configurations. However, these two averages factorize and can be performed in either order. Thus the velocity correlation function is

$$\langle\langle u(y(t_1))\cdots u(y(t_n))\rangle_w\rangle_c=\int_{-\infty}^{+\infty}dy_1dy_2\cdots dy_n\langle u(y_1)\cdots u(y_n)\rangle_c \times p(y_n,t_n)p(y_{n-1}-y_n,t_{n-1}-t_n)\cdots p(y_1-y_2,t_1-t_2)\,, \qquad (5)$$

where

$$p(x,t)=(1/\sqrt{4\pi Dt})\exp(-x^2/4Dt)$$

is the Gaussian probability distribution for the transverse motion. The product of Gaussians in Eq. (5) is the probability that a Brownian path visits the sequence of transverse positions $\{y(t_i)\}$ at times $\{t_i\}$, having started at $y=0$. For the continuous model defined by Eq. (1), $\langle u(y_1)\cdots u(y_n)\rangle_c$ is a sum of products of delta functions. Consequently, the second moment is

$$\langle\langle x(t)^2\rangle_w\rangle_c=2\sigma\int_0^t dt_1\int_0^{t_1}dt_2\int_{-\infty}^{+\infty}dy\,p(0,t_1-t_2)p(y,t_2)=\frac{4\sigma}{3\sqrt{\pi D}}t^{3/2}\,. \qquad (6)$$

Intriguingly, Eq. (6) does not fully characterize long-time transport properties, as there are anomalously large sample-to-sample fluctuations.[13] The longitudinal displacement, averaged over all walks in a *fixed* environment, $\langle x(t)\rangle_w$, depends on the configuration, and does not necessarily converge to zero at large times. However, the average over all environments, $\langle\langle x(t)\rangle_w\rangle_c$, does equal zero in the center-of-mass reference frame. Clearly $\langle x(t)\rangle_w$ has a distribution over environments which is a Gaussian of variance $\langle\langle x(t)\rangle_w^2\rangle_c$, which, for the continuous model of Eq. (1) is

$$\langle\langle x(t)\rangle_w^2\rangle_c=\sigma\int dy\langle\mathcal{N}(y,t)\rangle_w^2 =(\sqrt{2}-1)\frac{4\sigma}{3\sqrt{\pi D}}t^{3/2}\,. \qquad (7)$$

Thus both the configuration average of the mean-square displacement, $\langle\langle x(t)^2\rangle_w-\langle x(t)\rangle_w^2\rangle_c$, and the second moment $\langle\langle x(t)^2\rangle_w\rangle_c$ vary as $t^{3/2}$, but *with different prefactors.* This effect has been pointed out previously,[13] but actual values of these prefactors are obtained here for the first time.

An important consequence of these sample-specific fluctuations is that the probability distribution for the longitudinal displacement, $P(x,t)$, cannot reach a configuration-independent limiting form as a function of $x/t^{3/4}$, in a fixed environment. For a single "typical" environment, the typical width of $P(x,t)$ is presumably described by $\langle\langle x(t)^2\rangle_w-\langle x(t)\rangle_w^2\rangle_c$, while the configuration-averaged distribution can be viewed as the "envelope" of the individual diffusion fronts for each environment. The width of the latter is equal to

$$\langle\langle x(t)^2\rangle_w\rangle_c-\langle\langle x(t)\rangle_w\rangle_c^2=\langle\langle x(t)\rangle_w^2\rangle_c\,,$$

and is expected to be larger than the typical width, as found above.

In contrast to the individual $P(x,t)$'s for each environment, the configuration average is expected to take a well-defined scaling form in the large-time limit,

$$\langle P(x,t)\rangle_c\to t^{-3/4}f(x/t^{3/4})\,, \qquad (8)$$

where it is understood that x and t are simultaneously large with $u\equiv x/t^{3/4}$ finite (as is the case of any central-limit theorem). For $u\gg1$, the scaling function is expected to vary as

$$f(u)\sim\exp(-cu^\delta) \qquad (9)$$

(with possible power-law prefactors). According to our numerical and analytical arguments, the shape exponent δ appears to take on the anomalous value $\delta=\frac{4}{3}$.

Our numerical approach to test the validity of Eq. (9), and to find δ, is based on computing dimensionless moment ratios such as $m_{2k}(t)\equiv\langle x^{2k}(t)\rangle/\langle x^2(t)\rangle^k$ and

$$n_{2k}(t)\equiv\langle x^{2k}(t)\rangle/\langle x^{2(k-1)}(t)\rangle\langle x^2(t)\rangle .$$

If $\langle x^{2k}(t)\rangle\sim\langle x^2(t)\rangle^k$, the m_{2k} and n_{2k} will approach constants at $t\to\infty$ whose values depend on δ. By attempting to match our numerical estimates for m_{2k} and n_{2k} to the moments that arise directly from Eq. (9), we infer a value of δ.

We have developed several independent calculational approaches for these moments, all of which yield consistent results. One method is to evaluate Eqs. (4) and (5), thus yielding results which are *exact* for all times. This calculation becomes progressively unwieldy at higher order, and appears to be amenable only to numerical integration. Up to sixth order, we find $m_4\simeq 3.3\pm 0.03$ and $m_6\simeq 19.1\pm 0.4$, where the errors denote statistical uncertainties in numerical integration. These results indicate that $f(u)$ is non-Gaussian (a Gaussian gives $m_4=3$, $m_6=15$, etc.), but are insufficient to give a reliable estimate of δ.

A complementary numerical approach is based on using exact enumeration[2] to find the probability distribution of longitudinal displacements exactly for a given environment, and then averaging over *all* environments in a system of finite width w. For this computation, we employed the lattice model illustrated in Fig. 1(a). Each horizontal line is randomly assigned a velocity $\pm$, and at each lattice site a random walk moves either in the $+y$ or $-y$ direction with probability $\frac{1}{4}$ (thus fixing D), or moves with the bias with probability $\frac{1}{2}$. We performed this calculation for the 92 205 distinct velocity configurations (cyclic permutations and reflection symmetry), out of the 2^{23} states on a system of width $w=23$ with periodic transverse boundary conditions. This procedure provides the *exact* configurational-average probability distribution for an *infinite* system up to 22 time steps.

By extrapolation of the resulting moments to $t\to\infty$, a consistent trend in the behavior of n_{2k} as a function of k can now be discerned (Fig. 2). To extrapolate the moment values $n_{2k}(t)$, we form the sequence

$$i(t)=[\sqrt{t}\,n_{2k}(t)-\sqrt{t-1}\,n_{2k}(t-1)]/[\sqrt{t}-\sqrt{t-1}] ,$$

which are the intercepts at $1/\sqrt{t}=0$ of the straight line that passes through $n_{2k}(t)$ and $n_{2k}(t-1)$. This extrapolation is then repeated on the successive levels of extrapolant sequences. This approach yields a consistent picture in which the effective value of δ is a decreasing function k. Our estimate of n_4 suggests $\delta\approx 1.7$, while our estimate for n_8 is consistent with $\delta\lesssim 1.4$.

The apparent change as a function of the order of the moment suggests anomalous behavior of the tail of the distribution function.[14] Consider, therefore, the averaged probability of finding a "stretched out" walk, namely, $\langle P(x\sim t,t)\rangle_c$. According to Eq. (9), this probability should vary as $\exp(-t^{\delta/4})$. On the other hand, the probability of a walk being stretched out can be bounded from below by the probability of remaining transversely confined to a region of unidirectional velocity bias. This confining probability, averaged over all environments, is isomorphic to the survival probability of a one-dimensional random walk in the presence of randomly distributed traps,[15] and hence varies as $\exp(-at^{1/3})$. By comparing these two distributions, one concludes that $\delta\leq\frac{4}{3}$.

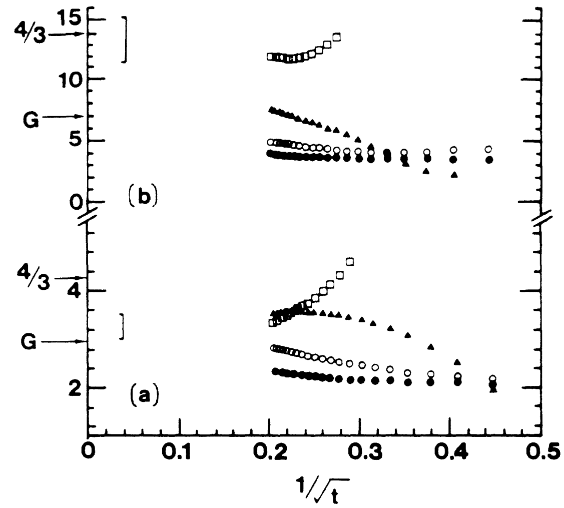

FIG. 2. Graphical estimates for the asymptotic behavior of (a) n_4 and (b) n_8. Plotted are the values for $n_{2k}(t)$ vs $1/\sqrt{t}$ (●), together with first (○), second (▲), and third (□) extrapolations based on plotting the intercepts at $1/\sqrt{t}=0$ of successive lower-order extrapolants. The arrows indicate the asymptotic values for $\delta=2$ (Gaussian) and $\delta=\frac{4}{3}$, while the square brackets indicate our subjective uncertainty estimates for n_{2k}.

To further argue that $\delta=\frac{4}{3}$, we first exploit the fact that the average in Eq. (4) can be first taken over environments for a fixed trajectory, and then over all walks. For a fixed trajectory, Eq. (2) expresses the longitudinal displacement as a sum of independent random variables, and it is thus clear that its distribution is a Gaussian, $\exp[-x^2/Q(t)]$, whose variance is proportional to $Q(t)\equiv\sum_y\mathcal{N}(y,t)^2$. While the complete distribution for $Q(t)$ is not readily calculable, it may be argued that the large-Q tail of this distribution is also a Gaussian. Consider first typical one-dimensional random walks which visit each site $\sqrt{t}$ times within a range $\sqrt{t}$. For such walks, $\langle Q(t)\rangle\sim(t/t^{1/2})^2t^{1/2}\sim t^{3/2}$. Now consider "confined" walks which fill a region of extent $w\sim t^\alpha$ with $\alpha<\frac{1}{2}$. We assume that these walks spread uniformly over the region t^α so that $Q(t)\sim(t/t^\alpha)^2t^\alpha\sim t^{2-\alpha}$. Consequently, for confined walks, $Q(t)/\langle Q(t)\rangle\sim t^{1/2-\alpha}$, while the probability of such walks varies as $\exp(-t/w^2) \sim \exp(t^{1-2\alpha})$. Therefore the distribution of $Q(t)$ has the Gaussian form, $P(Q) \sim \exp\{[-Q(t)/\langle Q(t)\rangle]^2\}$ for $Q \gg \langle Q(t)\rangle$. Now averaging the Gaussian displacement distribution $\exp[-x^2/Q(t)]$ over all random-walk trajectories, i.e., over $P(Q)$, leads to an averaged probability distribution of the form of Eq. (9), but with $\delta = \frac{4}{3}$.

An isotropic-random-velocity-field model [Fig. 1(b)] also exhibits superdiffusion. In two spatial dimensions, a walk moves on a random "Manhattan" grid, in which the directionality along any avenue or street is fixed along its entire length, but whose orientation is random. [This model can also be shown to be equivalent to a random walk in a divergenceless random velocity field with long-range correlations $\langle \mathbf{u}(0)\mathbf{u}(\mathbf{x})\rangle_c = |\mathbf{x}|^{-\alpha}$.[16]] For the random Manhattan system, we generalize the arguments of Eqs. (2) and (3) by formally decomposing the isotropic motion into transverse and longitudinal components. Assuming $x_{\rm rms} \sim t^{\nu}$, and then by following the steps that lead to Eq. (3), one finds that $x_{\rm rms} \sim t^{1-\nu/2}$. By isotropy, however, one must have $\nu = 1-\nu/2$, or $\nu = \frac{2}{3}$. Generalizing to arbitrary spatial dimension d yields $\nu = 2/(d+1)$ for $d < d_c = 3$, $\nu = \frac{1}{2}$ for $d > d_c$, and with logarithmic corrections for $d = d_c$. For the probability distribution of displacements, even modest simulations in two dimensions indicate that Eq. (9) holds over a substantial range, with $\nu = \frac{2}{3}$ and $\delta = 3$, in accord with the usual relation[17] between the shape and size exponent, $\delta = (1-\nu)^{-1}$.

In summary, superdiffusive transport arises from the interplay between pure diffusion and convection by spatially inhomogeneous, but correlated, velocity fields. For the layered system, we find a size exponent $\nu = \frac{3}{4}$ in two dimensions, with an anomalous large-distance tail in the averaged probability distribution of displacements. For an isotropic two-dimensional Manhattan system, the probability distribution appears to exhibit conventional scaling in which $\delta = (1-\nu)^{-1}$, and $\nu = 2/(d+1)$ for $d < 3$.

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