Lattice Calculation of the Kaon-Matrix-Element B Parameter

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We have calculated the kaon-matrix-element B parameter (B_K) using lattice QCD in the quenched approximation with staggered fermions. We find the correct chiral behavior. Errors from all sources except quenching are under control. At a lattice scale of 2 GeV our result is $B_K=0.70\pm0.01$(statistical) ±0.03(systematic). This translates to a renormalization-group-invariant value $\hat{B}_K=0.9-1.0$.

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It has now been 25 years since CP violation was first observed in the $\bar{K}K$ system. A long-standing theoretical challenge has been to deduce the consequences of this measurement for the parameters of the standard model. For this, one needs an accurate evaluation of a particular matrix element in quantum chromodynamics (QCD). We present here a calculation of this matrix element, using lattice QCD in the quenched approximation.

The CP-violating part of $\bar{K}K$ mixing is parametrized by the quantity ϵ. In the standard model ϵ is related to δ, the CP-violating angle in the Kobayashi-Maskawa (KM) mixing matrix, via

$$\epsilon=(\text{known factors})\times\sin\theta_{13}\sin\delta\, g(m_t)\hat{B}_K\,, \tag{1}$$

where θ_{13} is the usual b-to-u KM angle, and $g(m_t)$ is a known, monotonically increasing function of m_t.[1] The hadronic matrix element is parametrized by $\hat{B}_K$. Since ϵ is measured, knowledge of $\hat{B}_K$ would give a strong constraint on the KM matrix and m_t.

The matrix element needed to make the connection is

$$\mathcal{M}(\mu)=\langle\bar{K}\,|\,\bar{s}\gamma_\nu(1+\gamma_5)d\,\bar{s}\gamma_\nu(1+\gamma_5)d\,|\,K\rangle\,, \tag{2}$$

where μ indicates the scale at which the matrix element is evaluated. It is conventional to use the parametrization

$$\mathcal{M}(\mu)=\tfrac{16}{3}f_K^2m_K^2B_K(\mu)\,, \tag{3}$$

where the normalization is such that $f_K=113$ MeV, and all the nonperturbative QCD effects are lumped into the fudge factor $B_K(\mu)$. This choice is motivated by the vacuum-saturation approximation in which $B_K=1$, nominally at a scale $\mu\le1$ GeV such that $\alpha_s(\mu)\sim1$.[2] The quantity appearing in Eq. (1) is the combination

$$\hat{B}_K=B_K(\mu)\alpha_s(\mu)^{-2/9}\,, \tag{4}$$

which does not depend on the scale μ, as long as μ is large enough that perturbation theory is reliable.

Because it is a function of nonperturbative QCD physics, it is nontrivial to calculate B_K. Over the years various approximate methods have been applied to this problem. Lowest-order chiral perturbation theory gives $\hat{B}_K\sim0.33$,[1] QCD sum rules find values in the range 0.33–0.5,[3] while the large-N_c approximation yields 0.75(15).[4] Clearly there is considerable uncertainty in $\hat{B}_K$.

Lattice methods are well suited for a calculation of B_K. Nevertheless, it has proven difficult to carry through a reliable calculation. The first calculations were done with Wilson fermions,[5,6] which break the chiral symmetry of the continuum theory quite severely. This leads to extra systematic errors due to mixing with operators having different chiral transformation properties. We present here results of the first calculation using staggered fermions, for which the dangerous operator mixing is absent. Because of this, and other improvements, we are able to considerably reduce the errors.

Like the previous computations, this calculation is done using the quenched approximation, i.e., internal fermion loops are not included. This is a very significant truncation of the theory, the effects of which are not well understood. What is noteworthy about the present result is that we are able to reduce all the systematic errors from sources other than quenching to a low level.

Theoretical background.—A crucial check of lattice calculations is to reproduce the expected chiral behavior. If we vary the s and d quark masses, but keep both small, we expect

$$B_K=c_1+c_1'\frac{m_K^2}{(4\pi f_K)^2}\ln\left(\frac{m_K^2}{\Lambda^2}\right)+c_2\frac{m_K^2}{(4\pi f_K)^2}+O(m_K^4)\,, \tag{5}$$

where c_1, c_1', and c_2 are constants to be determined (Λ

can be absorbed into c_2). The constraint due to chiral symmetry is that $\mathcal{M}$ vanishes when $m_K^2 \to 0$, so that B_K tends to a constant. This constraint follows because the operator in $\mathcal{M}$ has a LL chiral structure. If one uses Wilson fermions, then chiral symmetry is explicitly broken, and mixing with LR operators is allowed. In this case $\mathcal{M}$ can tend to a nonzero value when $m_K^2 \to 0$. One has to try to subtract the LR component, which introduces extra systematic errors. With staggered fermions, enough chiral symmetry is maintained on the lattice to forbid mixing with LR operators, so that the lattice matrix element should have the correct chiral behavior without subtraction.[7,8] This result holds both in the quenched approximation and in the full theory.

Because of the finite volume of the lattice, we can only simulate quarks with masses down to about $m_s/3$. Thus our lattice "kaon" is composed of two almost degenerate quarks whose masses add up to m_s, in contrast to the physical kaon in which the d quark is almost massless. The leading coefficient c_1 is independent of the quark masses, and thus should be the same for both lattice and physical kaons. The nonleading coefficients c_1' and c_2 can, however, depend on the choice of quark masses. We can estimate this dependence by varying the lattice s and d masses within the range available to us.

In general, we do not expect the quenched approximation to yield the correct coefficients in Eq. (5). The only exception concerns the coefficient of the chiral logarithm, c_1'. This term is due to loops of pions, kaon, and η's, and is related by chiral symmetry to c_1. The ratio c_1'/c_1 is given in Table I for the different theories of interest. The table shows explicitly that c_1' depends on the quark masses, although the dependence is weak. More importantly, the quenched approximation gives results for the chiral logarithms which are very close to those in the full theory. This is not true for other quantities, for example f_K, for which the quenched calculation lacks the $m_K^2 \ln m_K^2$ terms present in the continuum.[10] It is only because the f_K dependence cancels in the ratio B_K that one gets the correct chiral logarithms.

The fact that the quenched approximation contains almost the correct chiral logarithms is important for two reasons. First, the chiral logarithms can make an important contribution to B_K. For example, if one takes the reasonable value $\Lambda \sim 0.8$ GeV, the chiral logarithm is $\sim 35\%$ of the leading term. Thus we might expect a quenched calculation of B_K to be more reliable than that of, for example, f_K.

The second reason is qualitative. Loops of light mesons are present in all calculations using full QCD, e.g., in the pion cloud surrounding the nucleon. For small pion masses, the loops involve propagation over long distances, and will be affected by the finite size of the lattice. Numerical calculations in full QCD will have to understand such effects, and B_K offers a first opportunity to study them.

The finite-volume dependence of B_K can be predicted because the ratio c_1'/c_1 is known. The calculation uses the methods of Ref. 11. The finite-volume shifts turn out to be small, less than 0.5% for all points in our data set. We have also calculated other matrix elements in which the predicted finite-volume dependence is larger.[12] In these cases our numerical results are consistent with the predictions, which gives us confidence in the prediction for B_K.

The arguments leading to Eq. (5) rely on the staggered-fermion flavor symmetry. This symmetry is partially broken at finite lattice spacing, leading to corrections to this equation. We can test for such corrections by comparing results at different lattice spacings.

The final issue is operator mixing. With staggered fermions, mixing with LR operators does not occur but one must still account for the perturbative renormalization of the lattice operators. Following the work of Daniel and Sheard,[13] we have completed this calculation.[14] We find that the corrections to B_K are less than a percent. There is, however, reason not to trust this result. Calculations of other quantities suggest that at $g \sim 1$, the one-loop perturbative estimates of corrections may be off by as much as a factor of 2. Thus we have decided to omit the perturbative corrections, and instead add 2% to our estimate of the systematic error. In this way we do not obscure our lattice results by choosing a particular implementation of the perturbative corrections.

Results.— Details of our computational methods are given in Refs. 15 and 16. The parameters of our ensemble of lattices and propagators are listed in Table II. The most time-consuming and reliable results are those obtained on the $24^3 \times 40$ lattices.

We show in Fig. 1 the results for B_K at $\beta = 6.0$ and 6.2. To convert the kaon masses to physical units we use the values for a^{-1} given in Table II. There is at least a 10% uncertainty in the estimate of a^{-1}, which ultimately feeds into a small (1.5%) systematic error in B_K. Statistical errors have been estimated using the jack-knife method. To obtain our estimate of the physically relevant B parameter, we fit the data with a linear function of m_K^2 and extract the value at the physical value of m_K^2. For $\beta = 6.0$ this involves interpolation, which we consider very reliable. For $\beta = 6.2$, the small physical size of the lattice limits us to larger masses, and we must extrapolate. This increases the errors somewhat. Our results for B_K at the physical kaon mass are collected in

TABLE I. Results for c_1'/c_1.

Theory	Masses	c_1'/c_1
Full[a]	$m_u = m_d = 0$	$-\frac{10}{3}$
Full[b]	$m_u = m_d = m_s$	-3
Quenched[b]	$m_u = m_d = 0$	-4
Quenched[b]	$m_u = m_d = m_s$	-3

[a]Reference 9. [b]Reference 10.

TABLE II. Lattice parameters and results for B_K.

β	a^{-1} (GeV)	$m_q a$	Size	Sample	B_K
5.7	1.0	0.005,0.01,0.015	$16^3\times32$	30	0.98 ± 0.02
6.0	2.0	0.03,0.02,0.01	$16^3\times40$	17	0.71 ± 0.02
				14	0.69 ± 0.03
				All	0.70 ± 0.02
			$24^3\times40$	15	0.70 ± 0.01
6.2	2.5	0.03,0.02,0.01,0.007	$18^3\times42$	27	0.69 ± 0.03

Table II.

The most important feature of Fig. 1 is that B_K appears to have a convergent limit as $m_K^2 \to 0$. Our results are thus consistent with the expected chiral behavior. Furthermore, the sum of the higher-order terms [those proportional to c_1' and c_2 in Eq. (5)] is small compared to the leading term at the physical kaon mass. The smooth extrapolation to $m_K^2=0$ is confirmed by results from the lattices at $\beta=5.7$. These are the largest in physical units (~3.2 fm across) thus allowing the use of the smallest kaon masses. Although the actual value of the amplitude is larger at $\beta=5.7$ (see Table II), we find no indication that the behavior of the amplitude changes at quark masses smaller than those shown in Fig. 1.

We have examined what happens when the s and d quarks are not degenerated by comparing, at $\beta=6.0$, results for $m_d a=0.01$, $m_s a=0.03$ with those for $m_s a=m_d a=0.02$. We find that the values of B_K agree within an error which is considerably smaller than the error in either measurement. If we extrapolate to the physical situation of an almost massless d quark the result for B_K will change by less than our statistical error, $\sim1\%$.

At $\beta=6.0$ we use lattices of two different sizes to test for finite-size effects. Figure 1 shows no evidence for such effects, so they must be smaller than the statistical errors. This is completely in accord with the theoretical expectations discussed above.

We have also checked that our statistical errors are reliable: We use two completely independent samples on the 16^3 lattices, and find that the results agree within their errors, as shown in Table II.

Since we have data at several values of β, we can say something about the approach to the continuum limit. We expect that B_K should have a mild dependence on a. Using Eq. (4) with the bare lattice couplings, we expect B_K to be $\sim1\%$ smaller at $\beta=6.2$ than at $\beta=6.0$. Our results are consistent with this expectation. The value of B_K at $\beta=5.7$ should be $\sim1\%$ larger than at $\beta=6.0$, while the change is in fact much larger. This implies that perturbation theory has broken down, probably due to large $O(a)$ corrections. Since we need perturbation theory to make contact with the real world via $\hat{B}_K$, results for $\beta<6.0$ are not trustworthy.

Conclusion.— We have a reliable quenched result for

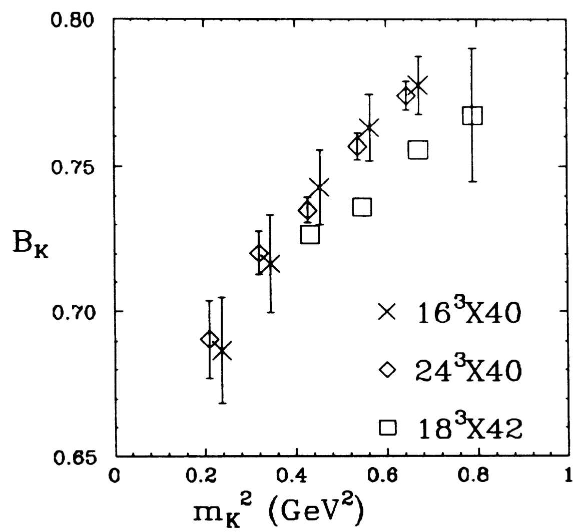

FIG. 1. B_K for $\beta=6.0$ and $\beta=6.2$ as a function of m_K^2 in physical units. The abscissas of the $\beta=6.0$ points have been symmetrically displaced, and the error on the $\beta=6.2$ has been included on only one point. The errors on the other points are similar.

B_K. The errors due to finite computer power—statistical and finite-volume errors—are under control, and are now comparable with errors from other sources, e.g., perturbation theory. For our final quenched answer we quote

$$B_K(2\text{ GeV})=0.70\pm0.01(\text{statistical})\pm0.03(\text{systematic}). \tag{6}$$

The systematic error reflects the effects of the lattice scale, finite volume, and perturbation theory. We stress that the systematic error due to our use of the quenched approximation is not known, and may be large.

Results with Wilson fermions are consistent with our value, though the former have a larger uncertainty. We quote the values for $B_K(2$ GeV$)$ at $\beta=6$. Using a $10^2\times20\times40$ lattice, the European Lattice Collaboration reports a value $B_K=0.81\pm0.16$,[5] where the error is only statistical. More recently, Bernard and Soni have used the same ensemble of quenched lattices employed in the present calculation, and obtain $B_K=0.83\pm0.11\pm0.11$ on the 16^3 lattice, and $B_K=0.66\pm0.08\pm0.04$ on the 24^3 lattice.[17] The first error is statistical, while the second is their estimate of the extra systematic error coming from the subtraction needed when using Wilson fermions. Systematic errors due to finite volume and lattice spacing are not included in these estimates. The fact that the Wilson and staggered results agree within their respective errors is an important check that the lattice systematics are not too severe.

To extract a physical number, and to compare with results from other approximation schemes, we use Eq. (4) to give a value for $\hat{B}_K$. Our final result is $\hat{B}_K=0.9$–1.0, where the uncertainty comes from the choice of $\alpha_s(\mu)$.

That the final number is close to the original vacuum-saturation value is quite ironic, especially considering that our calculation differs strongly from vacuum saturation at every intermediate step.

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