Exact Equivalence of Spin-$\frac{1}{2}$ Aharonov-Bohm and Aharonov-Casher Effects

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It is shown that there is an exact equivalence between the Aharonov-Bohm effect for spin-$\frac{1}{2}$ particles and the Aharonov-Casher effect. The demonstration of this precise relationship between the two is seen furthermore to be independent of whether relativistic or nonrelativistic kinematics are used. The only remaining substantive distinction between the two effects may well be the fact that the scattering cross section for polarized beams has a considerably greater structure in the Aharonov-Casher case despite the mathematical equivalence of the scattering amplitudes for the two effects.

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In 1959 it was shown by Aharonov and Bohm[1] (AB) that standard Schrödinger-equation analysis of the scattering of charged particles by a thin impenetrable solenoid implies the remarkable result that such particles are deflected even when classical forces are absent. Despite many attempts to disprove and/or deny the physical reality of this phenomenon, a growing body of experimental data has led to its increased acceptance as an observable effect. More recently, however, it has been observed by Aharonov and Casher[2] (AC) that the symmetries inherent in the Maxwell equations should imply results similar to those of the AB effect when magnetic dipoles are scattered from a filament carrying a uniform charge density. However, the correspondence they obtained was not an exact one. In particular, it required that the product of the magnetic moment times the electric field be much less than the mechanical momentum of the projectile. On the other hand, the electric field is neither constant nor bounded in the AC application. Furthermore, the required exclusion of slow particles cannot be readily accepted since nonrelativistic kinematics form the basis for the analysis of Ref. 2. Thus, while it is fairly clear that some kind of analogy exists between the two processes, it has never been stated precisely what the limitations on the correspondence between them happen to be.

This point is well illustrated, for example, by reference to an analysis of the AC effect by Boyer,[3] who attempted to explain it in terms of a classical lag. That interpretation in turn has been questioned by Aharonov, Pearle, and Vaidman.[4] However, the claim by Boyer that the AB effect somehow remains apart (and qualitatively different from the AC effect) by being immune to analysis in terms of a classical lag clearly indicates that an exact correspondence between the AB and AC effects has yet to be generally recognized. This view is also borne out by a recent work of Goldhaber,[5] who lists a specific set of differences between these two phenomena.

Before displaying the theorem which is the principal point of this paper it is well to revisit the derivation of Ref. 2. There is found that the relevant nonrelativistic Hamiltonian is

$$H_{\mathrm{NR}}=\frac{1}{2m}\boldsymbol{\sigma}\cdot(\mathbf{p}-i\mu\boldsymbol{\mathcal{E}})\boldsymbol{\sigma}\cdot(\mathbf{p}+i\mu\boldsymbol{\mathcal{E}}) \tag{1}$$

for a particle of mass m in an electric field $\mathcal{E}$. This is then recast into the form[6]

$$H_{\mathrm{NR}}=\frac{1}{2m}(\mathbf{p}-\boldsymbol{\mathcal{E}}\times\boldsymbol{\mu})^2-\frac{\mu^2\mathcal{E}^2}{2m}, \tag{2}$$

where $\boldsymbol{\mu}$ is given by $\boldsymbol{\mu}=\mu\boldsymbol{\sigma}$. However, the reduction of (1) to the form (2) is only possible if one drops a term proportional to $\nabla\cdot\boldsymbol{\mathcal{E}}$ which is, of course, proportional to the charge density of the source of the electric field. One may choose to argue that the particle is kept away from the interior of the charged filament by a potential which has yet to be introduced into the Hamiltonian. Such an approach would evidently be at least somewhat similar to the corresponding situation in the AB effect where the particle is required to remain exterior to the solenoid. Rather than dwell on this point, however, it is sufficient merely to remark that its significance will become evident in the equivalence proof to be given here.

In any event it is observed in Ref. 2 that for a radial $\mathcal{E}$ field the H_{NR} of Eq. (2) has the same formal structure as the Hamiltonian of a particle interacting with a thin solenoid provided that the quadratic term in $\mathcal{E}$ is neglected. The latter point has led to considerable misunderstanding, but can be readily dealt with here. To this end write the Hamiltonian (2) as

$$H_{\mathrm{NR}}=\frac{1}{2m}\sum_{i=1,2}[p_i-(\boldsymbol{\mathcal{E}}\times\boldsymbol{\mu})_i]^2+\frac{1}{2m}\{[p_3-(\boldsymbol{\mathcal{E}}\times\boldsymbol{\mu})_3]^2-\mu^2\mathcal{E}^2\} \tag{3}$$

and specialize to the case $\mathcal{E}_3=0$, $\partial_3\boldsymbol{\mathcal{E}}=0$ with H_{NR} restricted to the space of wave functions independent of the third (i.e., z) coordinate. This, of course, corresponds to the actual AC geometry in which $\mathcal{E}$ is the static electric field of a uniform line charge in the z direction and the particles are assumed to have no z component of momentum. In this case the last term in (3) vanishes and one has an equivalent Hamiltonian which is mathematically identical to that of the nonrelativistic AB effect for a spinless particle. Alternatively, had one started *ab initio* with a two-dimensional system the last term in (2) would never have been encountered and an identical conclusion would be obtained.

This strongly suggests that the correspondence between the AB and AC effects is considerably closer than was realized in Ref. 2. However, it is not yet clear as to whether this necessarily requires nonrelativistic kinematics or what the importance is of the neglect of the $\nabla\cdot\mathcal{E}$ term. These questions are most easily answered by carrying out an analysis of the AC effect within the framework of the Dirac equation. To this end one formulates the system in two space dimensions thereby implying the possibility of a two-component form for the spinor ψ. The Dirac equation for a neutral particle with magnetic moment μ can then be written as

$$\left[\gamma^{\mu}\frac{1}{i}\partial_{\mu}+m+\frac{\mu}{2}\sigma^{\alpha\beta}F_{\alpha\beta}\right]\psi=0,\quad \alpha,\beta=0,1,2\,, \tag{4}$$

where

$$\sigma^{\alpha\beta}=\frac{i}{2}[\gamma^{\alpha},\gamma^{\beta}]$$

and the Dirac matrices are taken to have the Pauli representation[7]

$$\beta=\gamma^0=\sigma_3\,,$$

$$\beta\gamma_i=\sigma_1,s\sigma_2,\quad i=1,2\,.$$

The parameter s can take the values $s=+1$ for spin "up" and $s=-1$ for spin "down." For the AC case (in which the only nonvanishing components of $F^{\mu\nu}$ are those of the electric field $\mathcal{E}^k=F^{0k}$) Eq. (4) reduces to

$$E\psi=\left[m\beta+\beta\boldsymbol{\gamma}\cdot\frac{1}{i}\nabla-i\mu\boldsymbol{\gamma}\cdot\boldsymbol{\mathcal{E}}\right]\psi\,,$$

which upon using

$$\gamma_i=is\epsilon_{ij}\beta\gamma_j\,,$$

with $\epsilon_{ij}=-\epsilon_{ji}$, $\epsilon^{12}=+1$, becomes

$$E\psi=\left[m\beta+\beta\boldsymbol{\gamma}\cdot\left(\frac{1}{i}\nabla-\mu s\tilde{\boldsymbol{\mathcal{E}}}\right)\right]\psi\,, \tag{5}$$

where $\tilde{\mathcal{E}}_i\equiv\epsilon_{ij}\mathcal{E}_j$. Equation (5) should be compared with the corresponding Dirac equation of a mass M particle in a vector potential $\mathbf{A}$,

$$E\psi=\left[M\beta+\beta\boldsymbol{\gamma}\cdot\left(\frac{1}{i}\nabla-e\mathbf{A}\right)\right]\psi\,,$$

i.e., the Dirac equation for the AB effect.[7] This leads to the immediate conclusion that the two effects are exactly equivalent (independent of whether relativistic or nonrelativistic kinematics are used) provided only that the vector potential and electric field are dual to each other, i.e.,

$$eA_i=\mu s\epsilon_{ij}\mathcal{E}_j\,. \tag{6}$$

Since the electric field in the usual AC effect (up to a normalization) is r_i/r^2 while the corresponding vector potential in the AB effect is $\epsilon_{ij}r_j/r^2$, where r_i is the two-dimensional radius vector, Eq. (6) is readily seen to be satisfied and the exact equivalence established.

Returning finally to the issue of the neglect of the $\nabla\cdot\mathcal{E}$ term in the analysis of Ref. 2, it should be remarked that in that work an attempt was made to establish equivalence to the spinless AB effect. Actually, the $\nabla\cdot\mathcal{E}$ term should properly be included and is easily seen to be the analog of the $\nabla\times\mathbf{A}$ term in the spin-$\frac{1}{2}$ AB equation. There it is identified as the Zeeman term which has been shown to have significant effects upon the scattering of polarized beams[7] as well as upon the equation of state of a system of flux-carrying fermions.[8]

Although the scattering amplitudes are equivalent in the spin-$\frac{1}{2}$ AB and AC effects when the identification (6) is made, the presence of the spin parameter s in that relation leads to a remarkable result for the scattering of polarized particles in the AC case. In Ref. 7 it was shown that the AB amplitude for the spin-$\frac{1}{2}$ case is

$$f(\phi)=-\left[\frac{i}{2\pi k}\right]^{1/2}\frac{\sin(\pi\alpha)e^{-iN(\phi-\pi)}}{\cos(\phi/2)}\times e^{-i\phi/2}\epsilon(\alpha s)e^{-i\phi\epsilon(s)\theta(-\alpha s)}\,, \tag{7}$$

where it is assumed that the incident beam is from the right and use has been made of the alternating function $\epsilon(x)\equiv x/|x|$ and $\theta(x)=\frac{1}{2}[1+\epsilon(x)]$. In (7) α is the flux parameter

$$\alpha=-\frac{e}{2\pi}\int\nabla\times\mathbf{A}\,d^2x\,,$$

with N being the largest integer contained in α. This result implies that for a beam polarized in the direction $\mathbf{n}$ with particles being detected whose polarization is along $\mathbf{n}'$ the cross section is

$$\frac{d\sigma}{d\phi}=\left(\frac{d\sigma}{d\phi}\right)_{\rm AB}\frac{1}{2}[1+(\mathbf{n}\cdot\hat{\mathbf{z}})(\mathbf{n}'\cdot\hat{\mathbf{z}})-(\mathbf{n}\times\hat{\mathbf{z}})\cdot(\mathbf{n}'\times\hat{\mathbf{z}})\cos\phi-\hat{\mathbf{z}}\cdot(\mathbf{n}\times\mathbf{n}')\sin\phi]\,,$$

where

$$\left(\frac{d\sigma}{d\phi}\right)_{\rm AB}=\frac{\sin^2(\pi\alpha)}{2\pi k\cos^2\phi/2}\,.$$

In the limit $\mathbf{n}=\mathbf{n}'$ this yields the result found in Ref. 7. For the AC effect with an electric field of the form

$$\mathcal{E}_i=2\lambda r_i/r^2\,,$$

it follows from (7) and the identification (6) that the scattering of a polarized beam in the AC case is

$$\frac{d\sigma}{d\phi}=\left(\frac{d\sigma}{d\phi}\right)_{\mathrm{AC}}\tfrac{1}{2}[1+(\mathbf{n}\cdot\hat{\mathbf{z}})(\mathbf{n}'\cdot\hat{\mathbf{z}})+(\mathbf{n}\times\hat{\mathbf{z}})\cdot(\mathbf{n}'\times\hat{\mathbf{z}})\cos(B\phi)+\hat{\mathbf{z}}\cdot(\mathbf{n}\times\mathbf{n}')\sin(B\phi)]\,, \tag{8}$$

with

$$\left(\frac{d\sigma}{d\phi}\right)_{\mathrm{AC}}=\frac{\sin^2(2\pi\mu\lambda)}{2\pi k\cos^2(\phi/2)}$$

and

$$B=2N+2-\epsilon(\mu\lambda)\,,$$

where N is now the largest integer in $2\lambda\mu$. The result (8) is striking in that (unlike the corresponding AB result) it displays a marked dependence on the parameter N. A sensitive experiment to detect the presence of this N-dependent term could provide one of the most significant possible tests of the entire methodology of the Aharonov-Bohm analysis.

It should be noted that the results given here make it possible to design experiments particularly well suited to displaying the qualitative differences between these two effects. One such possibility is realized by constraining the detector to accept only events for which the orientation of the polarization vector $\mathbf{n}'$ relative to the outgoing beam is identical to that of $\mathbf{n}$ with respect to the incident beam. For the AB case this yields

$$\frac{d\sigma}{d\phi}=\left(\frac{d\sigma}{d\phi}\right)_{\mathrm{AB}}, \tag{9}$$

while the AC result is

$$\frac{d\sigma}{d\phi}=\left(\frac{d\sigma}{d\phi}\right)_{\mathrm{AC}}\left[\sin^2\left(\frac{B-1}{2}\phi\right)+(\mathbf{n}\cdot\hat{\mathbf{z}})^2\cos^2\left(\frac{B-1}{2}\phi\right)\right]. \tag{10}$$

The fact that (9) has no residual polarization dependence may seem surprising initially, but is seen from (7) to be a consequence of the fact that the spin content of that equation is exactly given by the matrix for a rotation about the z axis through the scattering angle $\pi-\phi$. Such a result is readily understood when it is realized that the helicity operator $\mathbf{\Sigma}\cdot(\mathbf{p}-e\mathbf{A})$, where $\mathbf{\Sigma}$ is the spin operator commutes with the Hamiltonian. This observation leads in turn to the immediate result that the polarizations of the incoming and outgoing waves with respect to their directions are identical. The same analysis cannot be carried out in the AC case, however, because of the appearance of the spin factor in (6). This leads to the detailed dependence of the cross section on angle [as displayed in Eq. (10)] whenever the beam has a nonvanishing polarization component in the scattering plane.

In summary, then what has been established here is an exact equivalence between the AB and AC effects for relativistic spin-$\frac{1}{2}$ systems. On the other hand, it has also been shown that the solutions of these two-dimensional equations look rather different when examined in a three-dimensional context. It is to be hoped that the difficult experiments which can demonstrate these results will become feasible in the not too distance future.

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