Boundary Condition for Fluid Flow: Curved or Rough Surfaces

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(Received 11 December 1989)

The curvature of the boundary is shown to alter the fluid's slip length, which may even become negative as a result. As a result of the mesoscopic curvature of surface roughness, the microscopically calculated and the macroscopically measured slip lengths can be quite different.

PACS numbers: 47.45.Gx, 67.50.Dg

In this Letter, we consider the problem of fluid flow past solid walls. The boundary condition

$$v_\perp = \zeta v'_\perp \tag{1}$$

relates the tangential component $v_\perp$ of the velocity to its spatial derivative $v'_\perp$ normal to the surface. The slip length ζ is usually of the order of the mean free path λ. Neglecting it as one would in a hydrodynamic approach, we have $v_\perp = 0$ as the standard hydrodynamic boundary condition. This, however, is not necessarily true: Take the case of liquid ^{3}He that is contained in a vessel with walls covered[1] by superfluid ^{4}He. Here, the surface is very slippery, and without any roughness on a mesoscopic scale, the ^{3}He would slide past it with little friction. As a result, the slip length is macroscopic and much larger than the mean free path, $\zeta \gg \lambda$. Then the slip boundary condition (1) becomes genuinely hydrodynamic.

We raise three points in this Letter. First is the proof that Eq. (1) indeed represents the most general hydrodynamic boundary condition compatible with conservation laws and irreversible thermodynamics. Its form remains valid even if the surface is curved.

The second point is the fact that the slip length ζ has two contributions:

$$\frac{1}{\zeta} = \frac{1}{\zeta_0} - \frac{1}{R}\,. \tag{2}$$

The first term $1/\zeta_0$ is well known, microscopic in origin, and determined by the fraction of diffuse scattering off a flat boundary.[2,3] The second term, not considered before, is mesoscopic or macroscopic and simply given by the boundary's curvature radius R (along the projection of the velocity). R is taken to be positive for convex, and negative for concave, solid boundaries. This puts us into a somewhat new situation, especially when the solid surface retracts and R is positive. Then, in spite of diffuse scattering (i.e., a dissipating, rubbing wall or $\zeta_0 \neq \infty$), we may still encounter a diverging slip length ζ and a vanishing normal derivative $v'_\perp$. In fact, if $R < \zeta_0$, the total slip length may even become negative and the location of vanishing velocity goes into the liquid. After deriving Eq. (2), we calculate the velocity field between two concentric cylinders which rotate at different velocities. This example entails macroscopic curvature radii and illustrates nicely the physics of negative slip lengths.

The third point we raise is our main focus; it concerns mesoscopic curvature radii, more commonly referred to as surface roughness. There are a number of hydrodynamic experiments that are performed in the presence of *macroscopically* flat surfaces (one example is the impedance of an oscillating plate). They all provide information about a slip length. The usual prejudice in interpreting these experiments is that this slip length is just ζ_0, calculated *microscopically* assuming a flat surface. This has led to substantial disagreement between theory and experiment.[1] With the knowledge of Eq. (2), it is not hard to see why: The measured "effective" slip length ζ_{eff} contains the information of the appropriately averaged ζ of Eq. (2), which depends especially on the curvature radii of the surface roughness. Note that the existence of these curvatures, being mesoscopic, contradicts neither the microscopic nor the macroscopic flatness. To avoid misunderstanding, we close the introduction by emphasizing the following point. The contribution of the curvature radius, at each point of the surface, to the slip length ζ of Eq. (2) becomes important obviously only when ζ_0 is large. Nevertheless, the averaged, macroscopic slip length ζ_{eff} remains finite even if ζ_0 vanishes. In fact, ζ_{eff} is negative in this case; cf. Eq. (6) below. This can be interpreted as a shift of the surface position and, e.g., correctly accounts for the reduced total flux in a Poiseuille-flow experiment.

We start with the configuration of a two-dimensional flow in which the liquid is confined to the upper half space. Taking Δ to represent the difference of any quantity across the interface, the general *connecting conditions*[4,5] of shear flows between the two different hydrodynamic systems are (i) the equality of the transverse momentum currents perpendicular to the interface, $\Delta\Pi_\perp = 0$, and (ii) the proportionality of this current to the difference in the shear velocity, $\alpha\Pi_\perp = \Delta v_\perp$. Positivity of the entropy production requires $\alpha > 0$ (for a derivation cf. Ref. 5). Other pairs of analogous connecting conditions are also derived and studied in Ref. 5; e.g., (i) the equality of the entropy currents, $\Delta f = 0$, and (ii) $\kappa_s f = \Delta T$, with κ_s the Kapitza resistance and T the temperature. We define a slip length $\zeta_0 = \alpha\eta$, η being the shear viscosity contained in $\Pi_\perp$, and two vectors: $\hat{\mathbf{n}}$ is normal to the interface and pointing towards the liquid and $\hat{\mathbf{t}}$ is along $\mathbf{v}_\perp$. No third vector is needed, if we neglect the anisotropy of the surface material. The second connecting condition can be written as

$$\zeta_0(\nabla_k v_i + \nabla_i v_k) t_i n_k = v_i t_i . \qquad (3)$$

Note the following points: (i) This is the general *boundary condition* for shear flows in a fluid, since it solely involves the velocity of the liquid and its derivatives. (ii) It is valid only in the rest frame of the wall. (In fact, it is the Galilean invariance that yields $\Delta v_\perp = -v_\perp$ in the wall frame and renders the derivation easy. There is no equivalent way to relate ΔT to the temperature of the liquid alone.) (iii) With $\mathbf{v} = v_\parallel \hat{\mathbf{n}} + v_\perp \hat{\mathbf{t}}$ and $v_\parallel = 0$ at the boundary, the nonlinear term $\rho v_i v_k$ does not contribute to Eq. (3). (iv) Noting again the two formulas of (iii) and further that $(\hat{\mathbf{t}}\cdot\nabla)v_\parallel = t_i \nabla_k t_i = 0$, we find $t_i n_k \nabla_k v_i = n_k \nabla_k v_\perp$ and $t_i n_k \nabla_i v_k = v_\perp n_i (t_k \nabla_k) t_i$. Now, $t_k \nabla_k = d/ds$ and $dt_i = n_i\, d\phi$, where s denotes the distance traversed along the arc and ϕ the angle between $\hat{\mathbf{t}}$ and the horizontal line. Therefore, the second term can simply be written as $t_i n_k \nabla_i v_k = v_\perp / R$, where the curvature is $1/R = d\phi/ds$. We have $R > 0$ for the concave liquid (or convex solid) surface. Inserting these into Eq. (3) leads to Eqs. (1) and (2). These two are the physically transparent version, while Eq. (3) is frequently the computationally convenient one.

We proceed to consider the effect of macroscopic curvatures: Two concentric cylinders of radii $R_1 < R_2$, with a fluid in between, rotate at different angular velocities, ω_1 and ω_2, respectively. The smaller cylinder (index 1) has the slip length $(\zeta_0^{-1} + R_1^{-1})^{-1}$, while for the larger one it is $(\zeta_0^{-1} - R_2^{-1})^{-1}$. The resulting velocity field is purely azimuthal, $v = ar + br^{-1}$, with radial dependence. Denoting $A \equiv (1 - 2\zeta_0/R_2)/R_2^2$ and $B \equiv (1 + 2\zeta_0/R_1)/R_1^2$, we have

$$a = \frac{A\omega_1 - B\omega_2}{A - B}, \quad b = \frac{\omega_1 - \omega_2}{B - A} . \qquad (4)$$

So for $\zeta_0 = R_2$, we have $\partial v/\partial r = 0$ at R_2, and $\partial v/\partial r < 0$ for $\zeta_0 > R_2$. At first sight, this result seems puzzling. But there is really no cause for worry, since the physically relevant quantity, the torque (per unit area) transmitted from the inner to the outer cylinder, is a positive and monotonically decreasing function of ζ_0, vanishing with $\zeta_0 \to \infty$:

$$\int da\, |\mathbf{r} \times \Pi_\perp| = 2\eta \frac{\omega_1 - \omega_2}{(B - A)R_2} .$$

This experiment appears easy enough to perform, though it does require an exceedingly smooth surface. The reason is of course that surface roughness is always present, and one has to substitute ζ_{eff}, mentioned before, for ζ_0 in all the above formulas; cf. Eqs. (5)-(8) below. Then even with a generous layer of ^{4}He at the wall to achieve maximal specularity and slip ζ_0, a surface rough enough will still render ζ_{eff} much smaller than R_2. [It may be of interest to note that if one employs the usual boundary condition, Eq. (1) with $\zeta = \zeta_0$, the torque is positive only for $\zeta < R_1$ and unphysically negative for $\zeta > R_1$: a further sign, if one is needed, of the incompleteness of this boundary condition.]

We now turn to mesoscopic curvature radii. Our models for surface roughness are given by superpositions of sinusoidal waves, each with wave number k_i and amplitude h_i. Both have to be much larger than the mean free path λ, yet much smaller than any linear, macroscopic dimensions such as the smallest distance L in the apparatus or the viscous penetration depth δ. To calculate ζ_{eff}, the crucial point is the fact that in deriving Eq. (3), equivalently Eqs. (1) and (2), it was not necessary to specify the interface.[5] Hence we may, on the one hand, take Eq. (1) as the mesoscopic boundary condition to calculate the velocity field in the presence of a given surface irregularity. On the other hand, we are equally justified to take the interface as a broader entity, within which the roughness-induced modification of this velocity field becomes exponentially small. Then, we may again use Eq. (1) as the boundary condition, albeit a macroscopic one. It has a different slip length, which we denote as ζ_{eff}. Performing both calculations at once, we can equate the results to obtain ζ_{eff} as a function of ζ_0 and the parameters characterizing the surface roughness. The following is noteworthy: Since Eq. (1) is generally valid, and especially independent of the precise meaning of the interface, we may think of it as the structure of the boundary condition for shear flows.

Now we consider the specific case of Poiseuille flow between two parallel, rough plates of distance L, with an external pressure applied along the x direction. The linearized differential equations are $\nabla\cdot\mathbf{v} = 0$ and the stationary Navier-Stokes equation is $\nabla p = \eta\Delta\mathbf{v}$. Their solution must satisfy $v_\parallel = \mathbf{v}\cdot\hat{\mathbf{n}} = 0$ and Eq. (3), at the boundary defined by $y_\pm = L/2 \pm [L/2 - y_R(x)]$.

As a first example, we consider the case of a weakly varying surface of the form

$$y_R(x) = \sum_n [h_n^c \cos(nkx) + h_n^s \sin(nkx)] .$$

The velocity field is straightforward to find[6] and one can compute the total flux $Q = Q(y_R)$. This we equate, as explained above, with the flux between two macroscopically flat surfaces characterized by an effective slip length ζ_{eff}. The result is, up to order κ^2,

$$\zeta_{\text{eff}} = \frac{\zeta_0(1 - \frac{5}{4}\sum_n \kappa_n^2) - \sum_n \kappa_n^2/nk(1 + 2nk\zeta_0)}{1 + \sum_n \kappa_n^2(\frac{1}{2} + nk\zeta_0)} , \qquad (5)$$

where $\kappa_n = nk[(h_n^c)^2+(h_n^s)^2]^{1/2}$. Hence, as one would expect, because of the additional mechanism for momentum transfer, the macroscopic $\zeta_{\rm eff}$ is always smaller[7] than the microscopic ζ_0. In two limiting cases, simple addition laws hold: For the "stick" limit, $k\zeta_0 \ll 1$, $\zeta_{\rm eff}$ may be decomposed into two lengths according to

$$\zeta_{\rm eff} = \zeta_0 + \zeta_w^0 , \tag{6}$$

with a new temperature-independent length $\zeta_w^0 = -\sum_n \kappa_n^2/nk$ characterizing the surface roughness of the wall. This result can be easily interpreted as an effectively reduced cross-sectional area. If $k\zeta_0 \gg 1$, on the other hand, we find

$$\zeta_{\rm eff}^{-1} = \zeta_0^{-1} + \zeta_w^{\infty\,-1} , \tag{7}$$

with a new length $\zeta_w^\infty = (\sum_n nk\kappa_n^2)^{-1}$, again temperature independent.

Our second example is given by the superposition of two incommensurate sinusoidal curves $y_R(x) = h_1 \times\cos(k_1 x) + h_2\cos(k_2 x)$. The general expression[6] for $\zeta_{\rm eff}$ is quite similar to Eq. (5); one simply has to replace nk by k_n.

As a third example we consider the case of a simple wavy surface, $y_R(x) = h\sin(kx)$, but relax the assumption of weak variation $\kappa \ll 1$. We have solved this problem numerically. In addition, we obtained an interpolation formula for the effective slip length: As a function of $k\zeta_0$, $\zeta_{\rm eff}$ varies monotonically from the exact limiting value $\zeta_w^0(\kappa)$ to $\zeta_w^\infty(\kappa)$, to within a few percent deviation from the numerical result, as

$$\zeta_{\rm eff} = \frac{\kappa^2 k\zeta_0 \zeta_w^\infty(\kappa) + \zeta_w^0(\kappa)/(1+2k\zeta_0)}{1+\kappa^2 k\zeta_0} . \tag{8}$$

The input $\zeta_w^0(\kappa) = -w_0(\kappa)\kappa^2/k$ and $\zeta_w^\infty(\kappa) = w_\infty(\kappa)/k\kappa^2$ is provided numerically; see Fig. 1 (solid lines). The functions $w_{0,\infty}(\kappa)$ are found to decrease monotonically from 1, their value for the weakly varying limit. An expansion in small κ yields

$$w_0(\kappa) = \frac{1-\kappa^2/4+19\kappa^4/96+O(\kappa^6)}{1+\kappa^2(1-\kappa^2/2)/2+O(\kappa^6)}$$

and

$$w_\infty(\kappa) = \frac{1-5\kappa^2/4+61\kappa^4/64+O(\kappa^6)}{1+\kappa^2(1-5\kappa^2/8)/2+O(\kappa^6)} .$$

The result is represented by the broken lines in Fig. 1. For $\kappa \le 0.5$ the κ expansion is obviously an excellent approximation.

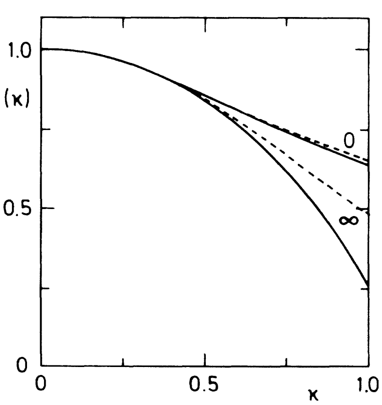

FIG. 1. Normalized asymptotic effective slip lengths $w_0(\kappa)$ and $w_\infty(\kappa)$ vs κ. Solid lines, exact numerical results; dashed lines, fourth-order κ expansions.

Although all three examples have yielded plausible and similar results for the contribution of surface roughness to the slip length, it seemed prudent to examine a different physical situation with these boundaries. If our concept is correct, Eqs. (1) and (2) are valid independent of the situation, and so neither should the slip length $\zeta_{\rm eff}$ depend on it. We consider the shear impedance $Z_\perp$ of a plate oscillating with a velocity $v = v_0 \times\exp(i\omega t)$. The complex impedance $Z_\perp = X+iY$ is defined as $\int da\,\Pi_\perp/v$ and can again be calculated either way: mesoscopically for a given wavy surface and macroscopically for a flat one characterized by an effective slip length. Reassuringly, the resulting slip length $\zeta_{\rm eff}$ for the low-frequency impedance of the boundary shape $y_R(x) = h\sin(kx)$ is the same[6] as given by Eq. (5).

Finally, we compare our results to the experimental data of Ritchie, Saunders, and Brewer.[1] At 40 mK, the measured torsional-oscillator impedance with 4% ^{4}He on the walls yields a ratio $Y/X \cong 0.6$ or $\zeta_{\rm eff} \cong \delta/3 \cong 5.3$ μm. ($\delta = (640\ \mu{\rm m})/[T/(1\ {\rm mK})]$ is the viscous penetration depth at saturated vapor pressure.) For simplicity, we assume total specularity, or $\zeta_0 \to \infty$. Then $k^{-1} = 1.8$ μm, $h = 1$ μm is a plausible pair of values that, via Eq. (8), leads to the right size for $\zeta_{\rm eff}$. They compare well with the measured roughness of 1 μm normal to the surface. The shear impedance $Z_{\perp p} = X_p + iY_p$ of pure ^{3}He (where $\zeta_0 \cong 0$) was also measured. As expected, its slip length $\zeta_{\rm eff}^p$ is close to zero, ζ_w^0 being a factor of 10 smaller than ζ_w^∞ for the above pair of values. With $\zeta_{\rm eff} = 6$ μm and $\zeta_{\rm eff}^p = 0$, we obtain $Y/Y_p = 0.50$, $X/X_p = 0.86$. They agree fairly with the measured values of 0.45 and 0.70, respectively. We must not extend this consideration to lower temperatures, since at 9 mK the mean free path $\lambda \cong (70\ \mu{\rm m})/[T/(1\ {\rm mK})]^2$ becomes equal to h, invalidating any calculations with the mesoscopic boundary condition. The macroscopic boundary condition, however, remains valid, since $\lambda \ll \delta$ for the entire experimental temperature range between 4 and 40 mK. So, although Eq. (8) cannot be expected to remain correct, $\zeta_{\rm eff} = 0.4\delta = (260\ \mu{\rm m})/[T/(1\ {\rm mK})]$ is what one can still conclude from the essentially temperature-independent ratios Y/X, Y/Y_p, and X/X_p.

We summarize as follows: Shear flows obey the slip boundary condition, Eq. (1), quite generally. Mesoscopically, the slip depends on both the specularity and the local curvature of the interface. Macroscopically, it also depends on the surface roughness. We have calculated the macroscopic effective slip length for two different physical situations, Poiseuille flow and the transverse surface impedance of an oscillating plate. As expected, the results are the same. Our models for surface roughness include periodic and nonperiodic ones. A more comprehensive account of our results will be published elsewhere.[6]

Sometime ago, H. Schmidt argued that boundary conditions are not always extrinsic add-ons to hydrodynamics. As an example, he offered the case of total slip on a wavy surface which he, as it now turns out, correctly thought should yield complete stick on a larger scale. We gratefully acknowledge that it was this comment that sent us along the line of work reported above.

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