Self-Dual Chern-Simons Vortices

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We study vortex solutions in an Abelian Chern-Simons theory with spontaneous symmetry breaking. We show that for a specific choice of the Higgs potential the vortex satisfies a set of Bogomol'nyi-type, or "self-duality," equations.

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Charged planar matter interacting with "photons" whose dynamics is governed not only by the Maxwell Lagrange density $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ but also by the Chern-Simons term $(\kappa/4)\epsilon^{\mu\nu\alpha}F_{\mu\nu}A_\alpha$ gives rise to topologically massive (2+1)-dimensional "electrodynamics," with gauge-field equations

$$\partial_\nu F^{\nu\mu}+(\kappa/2)\epsilon^{\mu\alpha\beta}F_{\alpha\beta}=J^\mu\,. \tag{1}$$

Here, $J^\mu=(\rho,\mathbf{J})$ is the conserved matter current and our Minkowski-space metric tensor $\eta_{\mu\nu}$ is diag$(1,-1,-1)$. The time component of Eq. (1) is the Chern-Simons-modified Gauss law

$$\nabla\cdot\mathbf{E}-\kappa B=\rho\,, \tag{2}$$

where $E^i=F^{i0}=-\dot{A}^i-\nabla_i A^0$, and $B=-F^{12}=\nabla\times\mathbf{A}$. Upon integration over the entire plane, this has the important consequence that any excitation with charge $Q=\int d^2r\,\rho(t,\mathbf{r})$ also carries magnetic flux $\Phi=\int d^2r \times B(t,\mathbf{r})$ given by

$$\Phi=-(1/\kappa)Q\,. \tag{3}$$

The first term in Eq. (2) integrates to zero owing to the long-distance damping effected by the "photon" mass $|\kappa|$: All gauge-invariant gauge-field quantities are short range. For the same reason, the spatial integral of B converges, but then it also follows necessarily that $\mathbf{A}$ is long range, so that the spatial integral of $\nabla\times\mathbf{A}$ is nonzero. Thus, charged systems also carry a vortexlike magnetic field.[1]

While the normal particle content of the gauge-field degrees of freedom is a single excitation with mass $|\kappa|$, this is modified by coupling to a U(1)-symmetry-breaking scalar field. The gauge field then acquires two propagating modes, with distinct masses differing by κ. Together with the Higgs-field degree of freedom, there are three propagating excitations with different masses.[2]

When the symmetry is broken, there also exist classical vortex solutions.[3] These are of the Nielsen-Olesen variety,[4] except that they are charged, as required by Eq. (3).[5]

In this Letter we return to the problem of charged vortices, but with gauge-field dynamics governed solely by the Chern-Simons term.[6] This "Chern-Simons electrodynamics" may be viewed as the $\kappa\rightarrow\infty$ limit of the topologically massive model, where the relation (3) holds locally in space. The truncation is physically sensible at large distances and low energies, where the lower-derivative Chern-Simons term dominates the higher-derivative Maxwell term. In this limit, and with the symmetry-breaking realization, one of the two gauge-field masses becomes infinite, so that the corresponding particle decouples from the low-energy spectrum, leaving one massive gauge degree of freedom as well as the Higgs mode.[7] Not unexpectedly, a charged-vortex solution still exists.[8] Moreover, we show that for a specific choice of the Higgs potential the vortex satisfies a set of Bogomol'nyi-type, or "self-duality," equations.

The Lagrange density for our model is

$$\mathcal{L}=|D_\mu\phi|^2+\tfrac{1}{4}\kappa\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma}-V(|\phi|)\,, \tag{4}$$

where $D_\mu\phi=(\partial_\mu+ieA_\mu)\phi$. We require that the Higgs potential $V(|\phi|)$ have a symmetry-breaking minimum at $|\phi|=v$ and that it contain only renormalizable interactions. If we further normalize so that $V(v)=0$, the potential must have the form

$$V(|\phi|)=\frac{\lambda^2}{4}[(|\phi|^2-v^2)^3+\alpha^2v^2(|\phi|^2-v^2)^2]\,. \tag{5}$$

When $\alpha^2\leq\frac{3}{2}$, there is also a symmetric minimum at $\phi=0$. For the asymmetric minimum to be a global minimum we need $\alpha^2\geq 1$. The self-duality equations emerge when $\lambda=2e^2/|\kappa|$ and $\alpha=1$.

The theory possesses two propagating modes. In the symmetry-breaking phase the Higgs mode has mass $m_H=\lambda\alpha v^2$, while the gauge-field excitation carries mass $m_A=2e^2v^2/|\kappa|$.

Here we are interested in time-independent vortex solutions to the field equations that approach the asymmetric vacuum at spatial infinity. These are stationary points of the energy, which for static configurations may be written as

$$E=\int d^2r[|\mathbf{D}\phi|^2-e^2A_0^2|\phi|^2+\kappa A_0B+V(|\phi|)]\,, \tag{6}$$

where $\mathbf{D}=\nabla-ie\mathbf{A}$. By varying with respect to A_0, we obtain the relation

$$A_0 = \frac{\kappa}{2e^2}\frac{B}{|\phi|^2}. \qquad (7)$$

Since the charge density of a static configuration is $\rho = -2e^2 A_0 |\phi|^2$, Eq. (3) is recovered by integrating both sides of (7) over all of space. A further relationship involving the flux is obtained by noting that to have finite energy $\mathbf{D}\phi$ must vanish at spatial infinity. If $|\phi|$ is nonvanishing there, it then follows that asymptotically $\mathbf{A} = -(i/e)\nabla \ln\phi$, and hence

$$\Phi = -\frac{i}{e}\int_{r=\infty} dl \cdot \nabla \ln\phi = \frac{2\pi n}{e}, \qquad (8)$$

where n is a topological invariant which takes only integer values.

Substitution of (7) into our expression for the energy yields

$$E = \int d^2r \left(|\mathbf{D}\phi|^2 + \frac{\kappa^2}{4e^2}\frac{B^2}{|\phi|^2} + V(|\phi|) \right). \qquad (9)$$

Variation of this gives a set of second-order static field equations. We leave the analysis of these for elsewhere, and specialize now to the case $\lambda = 2e^2/\kappa$ and $\alpha = 1$. With this choice the Higgs- and gauge-field masses are equal,

$$m_H = m_A = 2e^2 v^2/|\kappa| \equiv m,$$

and the symmetric and asymmetric vacua are degenerate. The energy may then be rewritten, after an integration by parts, as

$$E = \int d^2r \left(|(\mathbf{D}_1 \pm i\mathbf{D}_2)\phi|^2 + \left| \frac{\kappa}{2e}\phi^{-1}B \mp \frac{e^2}{\kappa}\phi^*(v^2 - |\phi|^2) \right|^2 \right) \pm ev^2\Phi. \qquad (10)$$

We thus have a lower bound on the energy

$$E \geq ev^2|\Phi| = 2\pi v^2 |n|, \qquad (11)$$

which is saturated by fields obeying the self-duality equations

$$\mathbf{D}_1\phi = \mp i\mathbf{D}_2\phi, \qquad (12)$$

$$eB = \pm \frac{m^2}{2}\frac{|\phi|^2}{v^2}\left(1 - \frac{|\phi|^2}{v^2}\right), \qquad (13)$$

where the upper (lower) sign corresponds to positive (negative) values of Φ and n.

If we define $\phi = vge^{i\omega}$, then Eq. (12) implies

$$eA^i = (\nabla_i \omega \pm \epsilon_{ij}\nabla_j \ln g). \qquad (14)$$

When substituted into Eq. (13), this gives

$$\nabla^2\chi = m^2 e^\chi (e^\chi - 1), \qquad (15)$$

where $\chi \equiv \ln g^2$. This equation can also be obtained by requiring that the positive, energylike functional[9]

$$\mathcal{E} = \int d^2r [\tfrac{1}{2}(\nabla\chi)^2 + \tfrac{1}{2}m^2(e^\chi - 1)^2] \qquad (16)$$

be stationary.

Let us examine the solutions with axial symmetry, corresponding to $|n|$ elementary vortices superimposed at the origin. By appropriate gauge transformation, such solutions can be brought into the form

$$\phi(r,\theta) = vg(r)e^{in\theta}, \qquad (17)$$

$$eA^i(r,\theta) = \epsilon_{ij}(\hat{r}^j/r)[a(r) - n]. \qquad (18)$$

In order that the fields be nonsingular at the origin, we must impose the boundary conditions $g(0) = 0$ and $a(0) = n$. The requirement that ϕ approach the asymmetric vacuum at large distance implies $g(\infty) = 1$, while finiteness of the energy leads to $a(\infty) = 0$.

For configurations of this form, the magnetic field is $B = -a'/er$ (with the prime denoting differentiation with respect to r) so that the flux is

$$\Phi = \frac{2\pi}{e}[a(0) - a(\infty)] = \frac{2\pi n}{e}, \qquad (19)$$

as expected. The angular momentum is obtained from the momentum density $\mathcal{P}$ via

$$\begin{aligned} J &= \int d^2r\, \mathbf{r} \times \mathcal{P} \\ &= -\int d^2r [D_0\phi^* \mathbf{r} \times \mathbf{D}\phi + D_0\phi \mathbf{r} \times (\mathbf{D}\phi)^*] \\ &= -\int d^2r (\kappa/e) B [\mathbf{r} \times \nabla \mathrm{Arg}(\phi) - e\mathbf{r} \times \mathbf{A}] \\ &= (\pi\kappa/e^2)\int dr (a^2)' \\ &= -(\pi\kappa/e^2) n^2, \end{aligned} \qquad (20)$$

where Eq. (7) has been used on the third line.[10]

Substitution of this *Ansatz* into Eqs. (12) and (13) gives

$$g' = \pm ag/r, \qquad (21)$$

$$a'/r = \pm \tfrac{1}{2}m^2 g^2(g^2 - 1). \qquad (22)$$

Since the solutions for n and $-n$ are related by the transformation $g \to g$, $b \to -b$, we consider only the case $n > 0$. At large r, where $\delta \equiv 1 - g \ll 1$, Eqs. (21) and (22) may then be approximated by the linear equations $\delta' = -a/r$ and $a' = -2m^2 r\delta$. The solution with appropriate behavior at infinity is

$$g = 1 - CK_0(mr), \quad a = CmrK_1(mr). \qquad (23)$$

To find the behavior at small r, we attempt a powerseries solution and obtain (for positive n)

$$g=A(mr)^n-\frac{A^3}{2(2n+2)^2}(mr)^{3n+2}+O((mr)^{5n+2}),$$

$$a=n-\frac{A^2}{2(2n+2)}(mr)^{2n+2} \quad (24)$$

$$+\frac{A^4}{2(4n+2)}(mr)^{4n+2}+O((mr)^{4n+4}).$$

The constant A is not determined by the behavior of the fields near the origin, but is instead fixed by requiring the proper behavior as $r\to\infty$. Thus, for a solution with A too large, g reaches unity at some finite value r_1, with $a(r_1)>0$; for all $r>r_1$, both g' and a' are positive, with g and a both growing without bound as $r\to\infty$. If, instead, A is too small, a becomes negative, at some value r_2, while g is still less than unity. For $r>r_2$, both g' and a' are negative and, asymptotically, $g\to 0$ while a tends toward a negative constant a_∞. The value of A separating these two regimes gives the vortex solution we seek.

To solve Eqs. (21) and (22) numerically, we choose an initial value of A and then integrate out from the origin until either $g>1$ or $a<0$. We then repeat the procedure with a new value of A, chosen to be larger or smaller depending on which condition terminated the previous integration. By successive iteration of this procedure we find that $A=0.3239$, 0.0389, and 0.0029 for $n=1$, 2, and 3, respectively,[11] and obtain the solutions shown in Fig. 1. In Fig. 2 we plot the magnetic field and the magnitude of the (purely radial) electric field for the $n=1$ solution.

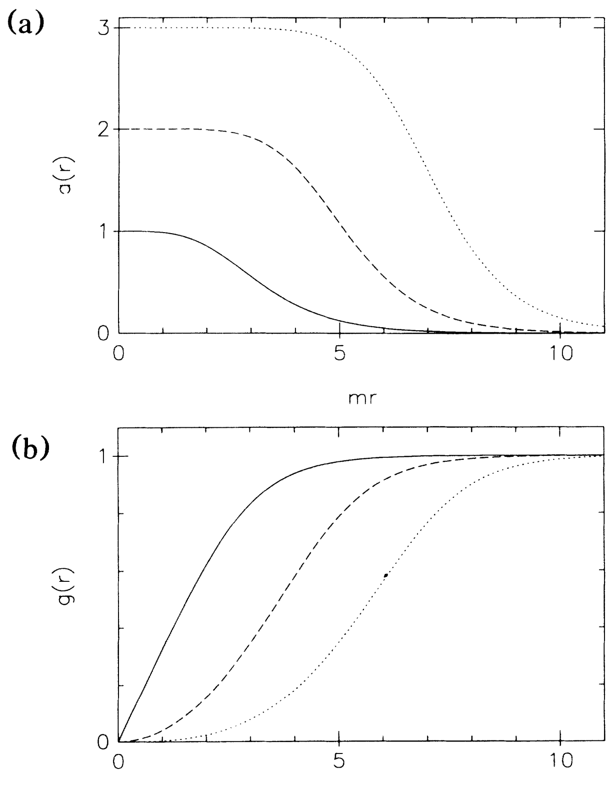

FIG. 1. The functions (a) $a(r)$ and (b) $g(r)$. The solid, dashed, and dotted lines correspond to $n=1$, 2, and 3, respectively.

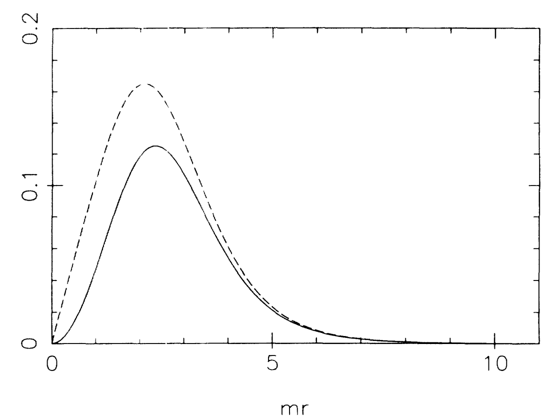

FIG. 2. The magnetic field (solid line) and the radial component of the electric field (dashed line) for the $n=1$ solution, in units of e/m^2.

Similar self-duality equations arise in the four-dimensional Ginzburg-Landau model with standard, rather than Chern-Simons, electromagnetism when the parameters are chosen to make the vector and scalar masses equal (i.e., when the model describes a system on the boundary between type-I and type-II superconductivity).[12] Two differences should be noted. First, while Eq. (12) still holds, Eq. (13) is replaced by one of the form $B\sim v^2-|\phi|^2$, so that the magnetic field is greatest at the center of the vortex. In contrast, the magnetic field for the Chern-Simons vortex is concentrated in a ring surrounding the zero of the Higgs field, with its maximum occurring when $|\phi|^2=v^2/2$. Second, in the Chern-Simons model equality of the Higgs- and gauge-field masses is a necessary, but not sufficient, condition for obtaining a self-dual system.

For the Ginzburg-Landau model, index-theorem methods can be used to show that any solution with topological charge n must depend on $2n$ continuous parameters, which may be understood as the positions of n noninteracting vortices.[13] The same methods can be used, with only very minor changes, to count the number of small perturbations which preserve the self-duality equations (12) and (13). One again finds that the general solution contains $2n$ parameters.

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Note added.—After completing this work, we learned that similar results have also been obtained by Hong, Kim, and Pac.[14]

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