Time-Dependent Solutions of 2+1 Gravity

Henri Waelbroeck
Center for Relativity, The University of Texas at Austin, Austin, Texas 78712
(Received 5 February 1990)

Time-dependent solutions of 2+1 gravity are found for N particles on a sphere with g handles. The theory is reduced to a constrained Hamiltonian system with a finite number of global observables. These include three-vectors that give the spacetime position of each particle, and others that are related to the Teichmüller parameters. With these global observables, the dynamical equations ($d^2x^i/dt^2=0$) decouple. The cases of the torus and a uniformly expanding sphere with N sources are shown to agree with solutions from the metric formulation.

PACS numbers: 04.20.Jb, 02.40.+m, 11.15.Ha

The formulation of 2+1 gravity in terms of a metric tensor field is known to be highly uneconomical, in the sense that of the continuous infinity of variables only a finite number are coordinate independent. Partly because of this, a large class of solutions of 2+1 gravity have yet to be found, particularly nonstatic spacetimes with spacelike slices of genus $g \geq 2$. In an elegant paper,[1] Witten suggested using instead a formulation of gravity on a reduced phase space which involves only coordinate-independent variables, identified the phase space as the moduli space of flat Poincaré connections, and offered some insights into its quantization.

The purpose of this Letter is to provide a set of variables for the phase space, a Hamiltonian, and solutions of the dynamical equations (a paper by Moncrief[2] based on the metric formulation provides a proof of existence, uniqueness, and smoothness of the Hamiltonian, and gives the Hamiltonian explicitly in the case of the torus, recovering in that case previous results by Martinec[3]). Further results, calculations, and proofs are postponed to a forthcoming publication.[4]

A lattice theory for 2+1 gravity with continuous time was developed by the author;[5] the phase-space variables are the link three-vectors expressed in local Minkowskian frames, and Lorentz matrices defining parallel transport between neighboring frames. The time evolution and gauge symmetries are generated by first-class constraints. These require that the curvature at each vacant lattice site vanish, relate the deficit angle at occupied sites to the mass of each particle, and require that the link vectors form closed faces. They act on the phase-space variables as generators of three-translations of each latticed site, translations of each particle along its world line, and Lorentz transformations of each frame.

The physical phase space is the set of solutions of the constraints modulo gauge transformations. A gauge-fixing procedure[4] leads to a reduced system with $12g+6N$ phase-space variables and $6+N$ constraints (this leaves $12g-12+4N$ observable phase-space variables; $4N$ to give the position and momentum of each particle on the surface, and $12g-12$ to parametrize the moduli space of flat Poincaré connections). Only one Minkowskian frame is kept, and all vacant lattice sites are translated to the same point which will be referred to as "the observer." The remaining constraints generate Poincaré transformations of the observer and his frame, and translations of each particle along its world line.

The variables and constraints of the reduced system are best understood in terms of an imaginary toy. Consider a genus-g surface with N points to represent the sources and one more point (O) that will be the observer. A two-dimensional boy who lives on this surface has just received a two-dimensional balloon for his birthday. He blows in his balloon, and blows, and blows, until finally he can blow no more. The balloon has wrapped itself around the surface, getting stuck at each of the $N+1$ points, until all of the "volume" of space is inside (Fig. 1). Note that the inside of the balloon has a topology of a disk, so it is simply connected. Now, since in 2+1 gravity the curvature tensor vanishes, any vector which is parallel-transported around a contractible loop will return unchanged, so spacetime within that region is flat. In other words the inside of the balloon as it evolves in time cuts a tube out of Minkowski space (Fig. 2). The adjacent segments of the balloon in Fig. 1 correspond to segments on the boundary of the Minkowskian tube that should be identified.

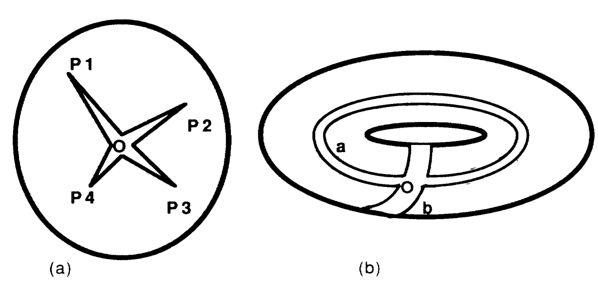

FIG. 1. Maximal disk for (a) the sphere with four particles and (b) the torus. The shaded area is the inside of the "balloon;" the white area shrinks to zero when the balloon is fully blown.

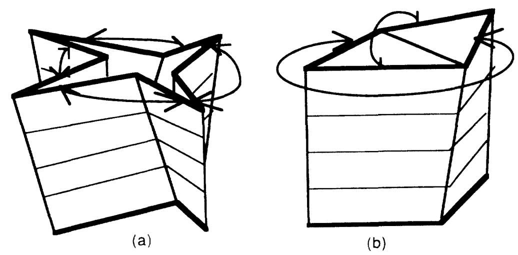

FIG. 2. "Minkowskian tube" representation of spacetime for (a) the sphere with four particles and (b) the torus. Arrows indicate which edges are identified.

The gauge-fixed lattice theory leads to precisely this picture, where the segments are represented by three-vectors [the remnants of the link vectors $E^a(ij)$ and $E^a(ji)$ of the lattice theory]. Two segments which need to be identified are related by a Lorentz transformation $\mathbf{M}$; this corresponds (Fig. 1) to parallel transporting across the pair of adjacent segments $[M^a{}_b E^b(ji) = -E^a(ij)]$. The constraints are that the curvature at the observer (O) vanishes, that the mass of each particle is a given function of the deficit angle, and that the balloon does not rip, hence the following reduced system [$\mu=1,\ldots,2g+N$; $-\mathbf{M}(\mu)\mathbf{E}(\mu)$ is identified with $\mathbf{E}(-\mu)$; the brackets are inherited from the lattice theory]: phase space,

$$[E^a(\mu),E^b(\mu)]=\epsilon^{abc}E_c(\mu)\,, \tag{1}$$

$$[E^a(\mu),M^b{}_c(\mu)]=\epsilon^{abd}M_{dc}(\mu)\,, \tag{2}$$

$$[E^a(-\mu),M^b{}_c(\mu)]=-M^b{}_d(\mu)\epsilon^{ad}{}_c\,; \tag{3}$$

constraints,

$$J^a=\sum_{(\mu)}[E^a(\mu)+E^a(-\mu)]\approx 0\,, \tag{4}$$

$$P^a=\epsilon^{abc}\left(\prod_{(\mu)}\mathbf{M}(\mu)\right)_{cb}\approx 0\,, \tag{5}$$

$$H(\mu)=3-\mathrm{Tr}[\mathbf{M}^2(\mu)]+4\sin^2[8\pi Gm(\mu)]\approx 0\,. \tag{6}$$

The order in which the $2g+N$ matrices in (5) are multiplied is obtained from Fig. 1 by drawing a loop around the observer and taking the product of Lorentz matrices as the loop crosses each pair of matched segments.

The brackets of the constraints in Eqs. (4)-(6) have the algebra of the Poincaré group. The constraints J^a act on the variables $\mathbf{E}(\mu)$ and $\mathbf{M}(\mu)$ as generators of SO(2,1) transformations. The action of the translation generators depends on the order of the matrices in (5); note that time evolution is encoded in these constraints since they generate translations of the observer also in the time direction. In general, the Hamiltonian is a linear combination of the constraints with arbitrary parameters which reflect the freedom to choose freely the position of the observer on the surface, the time parameters, and the orientation of the 2+1 frame in which the variables are expressed. If one takes a fixed observer and a nonrotating frame, the Hamiltonian becomes simply $H=P^0+\sum_\mu N(\mu)H(\mu)$. Since the brackets of $\mathbf{M}$'s among themselves vanish and the Hamiltonian in this gauge depends only on $\mathbf{M}(\mu)$, these are constants of the motion. Also, $(d/dt)E^a(\mu)=[H,E^a(\mu)]$ is a function of $\mathbf{M}$'s only, so the vectors $E^a(\mu)$ have zero acceleration:

$$(d^2/dt^2)E^a(\mu)=0\,. \tag{7}$$

The spacetime corresponding to this solution is a tube cut out of Minkowski space by a one-parameter family of polygons (the two-dimensional balloon at various times), with the edges identified two by two and such that the corners of the polygon lie on straight world lines (Fig. 2). The fact that the world lines must be straight can be understood, in a handwaving manner, by the fact that the spacetime curvature must vanish everywhere but at the sources. It clearly vanishes inside the "Minkowskian tube;" the requirement that spacetime is flat also at the edges can be satisfied only if the walls forming the boundary of the tube are flat.

Some aspects of these spacetimes tend to hide in the Minkowskian tube picture and should be pointed out. For instance, the velocities $d\mathbf{E}(\mu)/dt$ are not all independent sinced they are given by the brackets with the Hamiltonian in terms of the constrained matrices $\mathbf{M}(\mu)$. Also the "constant-time" slice given by the three-vectors $\mathbf{E}(\mu)$ is generally not planar—insisting that these surfaces be planar is *not* a choice of gauge, but a severe restriction on the matrices $\mathbf{M}(\mu)$, or on the velocities $d\mathbf{E}(\mu)/dt$.

Writing the solution in the form $g_{ij}(t)$ requires filling each polygon with a surface, picking coordinates on that surface $[x^a(\beta^1,\beta^2)]$, and computing the metric $g_{ij}=\eta_{ab}\times(\partial x^a/\partial\beta^i)(\partial x^b/\partial\beta^j)$.

Before turning to specific examples, the stage is now set to discuss the validity of this reduced system for continuum 2+1 gravity. A first point is that the reduced system [(1)-(6)] is completely independent of the number of points in the original lattice. Consequently, a renormalization-group transformation for the lattice theory can be found which cuts the link lengths by half at each step and leaves the reduced system unchanged. Furthermore, in the limit of very small links the action from which the lattice theory can be derived becomes the Chern-Simons action for 2+1 gravity, and the lattice action is a C^∞ function of the link variables.[5] Third, the number of observables $(6g-6+2N)$ is the same as for continuum 2+1 gravity, and the solutions are C^∞ functions of time. These facts (reduced system invariant under the renormalization-group, uv limit = continuum action, C^∞ action and dynamical behavior, and same number of solutions) are sufficient to prove that the solutions of 2+1 gravity are solutions of the lattice theory and vice versa.[4]

The variables of the reduced system can be expressed in terms of their continuum counterparts. The matrices $\mathbf{M}(\mu)$ are the Wilson-loop matrices which define parallel transport around the a loops and b loops of the surface, and around the pointlike sources. The vectors $\mathbf{E}(\mu)$ are path integrals of the dreibein around the b loops and a loops, respectively, and along paths that connect the observer to the various pointlike sources,

$$E^a(\mu)=\int_{(\mu)} e_i{}^a ds^i . \tag{8}$$

(The index a is parallel-transported back to the observer along the path.) An important role of the lattice theory is to provide a "regularization" of the brackets $[E^a(\mu), E^b(\mu)]$.

The torus is a somewhat peculiar case. The reduced system is described by a pair of vectors and matrices corresponding to the two noncontractible loops. The constraints are

$$J^a=E^a(1)-E^b(1)M_b{}^a(1)+E^a(2) -E^b(2)M_b{}^a(2)\approx 0 , \tag{9}$$

$$P^a=\epsilon^{abc}[\mathbf{M}(1)\mathbf{M}(2)\mathbf{M}^{-1}(1)\mathbf{M}^{-1}(2)]_{cb}\approx 0 , \tag{10}$$

where $\mathbf{M}(1)$ is the Wilson-loop matrix for the b loop which intersects the a loop corresponding to $\mathbf{E}(1)$, etc. The translation constraints are equivalent to the requirement that $\mathbf{M}(1)$ and $\mathbf{M}(2)$ commute, i.e., that their "rotation vectors" be parallel; this is only *two* independent conditions. There is also a redundancy among the constraints (9), which is that J^a is automatically orthogonal to the rotation vector. Thus of the six constraints only four are independent, and the number of degrees of freedom is $6-4=2$ (the dimension of Teichmüller space).

A solution of 2+1 gravity with spacelike slices having the topology of a torus is represented in Fig. 2(b). Each spacelike slice of this spacetime can be chosen to be a pair of triangles combining to form a four-edged surface in Minkowski space. To find a convenient pair of coordinates for this surface, consider the length-preserving map from the two-triangle surface to a parallelogram in R^2 with opposite edges identified, which can in turn be mapped with a linear transformation into a square with two coordinates (β^1,β^2) ranging from zero to 2π. The metric for these coordinates is given in terms of the lengths $l(1),l(2)$ of the two vectors $\mathbf{E}(1),\mathbf{E}(2)$ and their scalar product $s(12)$ as follows: $g_{11}=l^2(1)/(2\pi)^2$, $g_{12}=s(12)/(2\pi)^2$, etc. (To obtain the Teichmüller parameters, one may choose a gauge where the figure is a parallelogram of unit height; this generally requires having a moving observer, or nonzero "shift function."[4])

The solution of the dynamical equations, $\mathbf{E}(\mu)=\mathbf{A}(\mu)+\mathbf{B}(\mu)t$, leads to quadratic functions of time for $g_{ij}(t)$, for instance,

$$g_{11}=[\mathbf{A}^2(1)+\mathbf{A}(1)\cdot\mathbf{B}(1)t+\mathbf{B}^2(1)t^2]/(2\pi)^2 . \tag{11}$$

The fact that the torus admits globally defined coordinates in terms of which the metric is spatially constant makes it straightforward to solve Einstein's equations directly.[3] Taking, for instance, the metric $ds^2=-dt^2+g_{11}(t)dx^2+g_{22}(t)dy^2$, one finds that the Riemann tensor vanishes if $g_{11}(t)=(a+bt)^2$ and g_{22} is a constant, or vice versa. This solution agrees with (11) if $\mathbf{A}(1)$ is parallel to $\mathbf{B}(1)$ and $\mathbf{B}(2)=0$; the conditions on $\mathbf{A}(\mu)$, $\mathbf{B}(\mu)$ are required[4] by the constraints (9) and (10) together with the condition $g_{12}=0$.

Higher-genus surfaces do not admit a globally defined, constant metric, so the traditional approach becomes rather difficult. Perhaps the main result of this paper is that one should not interpret this difficulty as a sign that the dynamics become tremendously complicated for the higher-genus case, but rather that the metric variables become tremendously inconvenient.

Turning now to the sphere with N massive particles, the vectors $\mathbf{E}(\mu)$ indicate the spacetime position of each particle [Fig. 1(a)]. Each matrix gives the Lorentz transformation which is necessary to match the identified edges, and the constraints are

$$J^a=\sum E^a(\mu)-E^b(\mu)M_b{}^a(\mu)\approx 0 , \tag{12}$$

$$P^a=\epsilon^{abc}[\mathbf{M}(1)\mathbf{M}(2)\mathbf{M}(3)\cdots\mathbf{M}(N)]_{cb}\approx 0 , \tag{13}$$

$$H(\mu)=3-\mathrm{Tr}[\mathbf{M}^2(\mu)]+4\sin^2[8\pi Gm(\mu)]\approx 0 . \tag{14}$$

The solution is the spacetime corresponding to a tube cut out of Minkowski space by $2N$-sided polygons with adjacent sides identified two by two [Fig. 2(a)]. This describes the motion of N massive particles as straight world lines with a three-velocity parallel to $P^a(\mu)$ [the constraint (14) is $P^2(\mu)+\cdots\approx 0$ and $P^a(\mu)$ generates translations of the point μ]. Note that the polygon in Fig. 1(a) can be planar only when the "rotations" $\mathbf{M}(\mu)$ all have the same axis, in which case the particles' world lines are parallel.

There is no complete time-dependent solution for this problem already available in the literature; however, it is a straightforward task to introduce a time dependence in the static solution of Deser, Jackiw, and 't Hooft.[6] One finds

$$ds^2=-dt^2+\prod_{(\mu)}|r-r_\mu|^{-8Gm_\mu}(1+vt)^2(dx^2+dy^2) . \tag{15}$$

This is a uniformly expanding universe, where the geodesic distance between any two particles is a linear function of time. In the notation of this paper, this spacetime would be described by a Minkowskian tube with constant-time slices which are regular polygons, marked by $2N$ corners which lie on straight world lines diverging from a common origin.

To summarize, a new class of time-dependent solutions of 2+1 gravity with nontrivial topology has been presented. A key step is the use of a lattice theory to provide brackets for the global variables $E^a(\mu)=\int e_i{}^a \times ds^i$, for which the dynamical equations decouple. The validity of the lattice approach is justified by the fact that the reduced system is independent of how fine the lattice is, and that in the limit of very small links the lattice theory becomes ordinary $2+1$ gravity.

It is, in principle, possible to put a solution in the form of a Lorentzian metric on the chosen manifold. The principal difficulty in doing so resides in finding a convenient set of coordinates for each spatial slice such that the matching conditions are time independent. Given such coordinates and a time parameter, one expresses them in terms of the coordinates x,y,t of the Minkowskian tube and computes the Lorentzian metric from the Minkowski metric with a coordinate change; the curvature tensor is zero by construction.

The solutions presented in this paper can be used to generate solutions of $3+1$ gravity by adding a trivial fourth dimension to any of the spacetimes above, leading to a spacetime $M^3 \times R$ which automatically solves Einstein's equations since the curvature tensor vanishes everywhere except at the sources (which have become cosmic strings[7,8]).

Among many points which have not been addressed in this Letter are the following.[4] (1) One should require that the variables $\mathbf{E}(\mu)$ describe a spacelike surface; however, depending on the initial velocities part of the surface may become lightlike in a finite amount of time, at which point the solution becomes unphysical [and $\det(g)=0$]. (2) The Minkowskian tube picture breaks down when a corner is on a collision course with one of the walls; this requires switching to a new set of variables where the observer is linked to the various particles via different paths. (3) How would astronomers notice if spacetime did in fact have regions that are sufficiently close to the form $M^3 \times R$ for the solutions described in this paper to apply?

I would like to thank Sergio Hojman and Professor Richard Matzner for many enlightening discussions. This work is supported in part by NSF Grant No. PHY8806567.

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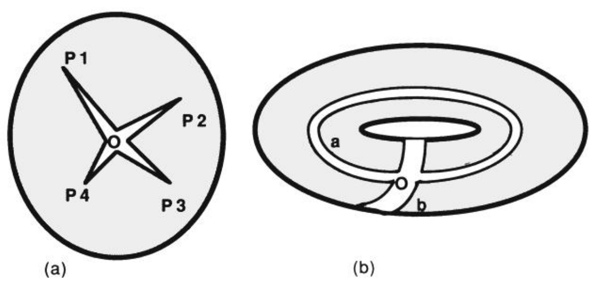

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