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"Quantum" Chaos in Billiards Studied by Microwave Absorption

H.-J. Stöckmann and J. Stein

Fachbereich Physik, Universität Marburg, D-3550 Marburg, West Germany

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The eigenfrequencies of resonance cavities shaped as stadium or Sinai billiards are determined by microwave absorption. In the applied frequency range 0–18.74 GHz the used cavities can be considered as two dimensional. For this case quantum-mechanical and electromagnetic boundary conditions are equivalent, and the resonance spectrum of the cavity is, if properly normalized, identical with the quantum-mechanical eigenvalue spectrum. Spectra, containing up to a thousand eigenfrequencies, are obtained within minutes. Statistical properties of the spectra as well as their correlation with classical periodic orbits are discussed.

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The statistical properties of the eigenvalue spectrum of a Hamiltonian of a quantum system change in a characteristic way if the corresponding classical system shows a transition from integrable to nonintegrable behavior. Whereas in integrable systems for the distribution of energy differences of successive eigenenergies Poisson statistics is observed, the spectra of nonintegrable systems obey Wigner statistics. Experimentally, Wigner statistics were already observed many years ago in the spectra of highly excited nuclei,[1] a very recent example is given by the optical fluorescence spectra of NO_2 molecules.[2] Intense theoretical studies exist for the periodically kicked top,[3,4] nonlinearly coupled oscillators,[5,6] and billiards of different shapes.[7,8] Pechukas[9] and Yukawa[10] showed that the strength of the nonintegrable part of the Hamiltonian may be interpreted as a pseudotime. As a function of this "time," the eigenenergies move on the energy axis in a similar way as the particles of an interacting one-dimensional gas.

An alternative approach to the understanding of the statistics of eigenvalues is opened by the semiclassical approximation. Gutzwiller[11] showed that the density of eigenvalues $\rho(E)$ can be decomposed into a monotonic and an oscillatory part,

$$\rho(E) = \rho_0(E) + \sum_\gamma \rho_\gamma \cos[(1/\hbar)S_\gamma - \phi_\gamma] \qquad (1)$$

(see also Ref. 12 for a review). In the case of two-dimensional billiards, the monotonic part $\rho_0(E)$ is given by[13]

$$\rho_0(E) = \frac{A}{4\pi} - \frac{L}{8\pi}\frac{1}{\sqrt{E}} + \cdots, \qquad (2)$$

where A is the area and L is the circumference of the billiard (the units were chosen such that $\hbar^2/2m = 1$, i.e., $E = k^2$, where k is the de Broglie wave number of the particle). The oscillatory part of $\rho(E)$ in Eq. (1) is a sum over all classical periodic orbits γ. S_γ is the classical action for the orbit. The prefactor ρ_γ and the phase ϕ_γ can be calculated within the frame of the semiclassical approximation. If the action S_γ is proportional to a power of E, then the contributions from the different periodic orbits to $\rho(E)$ can be obtained by a Fourier transformation of the spectrum as was demonstrated by Wintgen.[14] The classical periodic orbits show further up in so-called "scars," regions of extra high amplitudes of some eigenfunctions in the neighborhood of periodic orbits.[15,16]

Calculations of eigenvalues of nonintegrable Hamiltonians are extremely time consuming even on modern computers. Probably for this reason in all publications mentioned above the total number of eigenvalues calculated for one system was of the order of at most several hundred. Parameter dependences have been studied only over very limited regions. In billiards only two-dimensional systems have been studied up to now. Information on the eigenfunctions is even more scarce.

It is shown in this Letter that even in a time of rapidly growing computer capabilities an experimental approach to the problem may be an attractive alternative. Our experiment uses the fact that the time-independent Schrödinger equation and the wave equation are mathematically equivalent (apart from the boundary conditions which may be different). The billiards are substituted by suitably shaped resonators, whose eigenfrequencies are measured. There are several alternatives for an experimental realization, such as vibrating plates, optical resonators, and microwave resonators. Several authors[17,18] proposed the use of optical waveguides to simulate kicked quantum systems. For the simulation of billiards the use of microwave cavities seemed most promising to us for the following reasons: (i) Microwave generators for a wide frequency range in the GHz region and network analyzers for the registration of eigenfrequencies are commercially available, (ii) the construction of cavities with sizes in the cm range is easy and inexpensive, and (iii) it is no problem to obtain resonator qualities Q of the order of several thousands. With a somewhat greater effort even qualities of several ten thousands are obtainable. Interestingly enough, the same approach was used already 35 years ago to mimic the *acoustic* properties of rooms with the help of microwave cavities.[19]

The resonator quality Q in a frequency range of interest is defined as $Q = \nu/\Delta\nu$, where ν is an eigenfrequency in that range and $\Delta\nu$ is its typical width. The main cause for the width is the loss of microwave energy in the walls of the resonator due to the skin effect. For microwave frequencies of about 10 GHz typical skin depths for a good metallic conductor such as brass, which was used in the present experiments, are of the order of 1.5 μm. Two successive eigenfrequencies can just be resolved, if they are separated by at least $\Delta\nu$. A simple calculation shows that the total number of eigenfrequencies which can be registered is of the order of Q.

In our experiments we use a Hewlett-Packard microwave generator HP 8350 B together with a scalar network analyzer HP 8757 A. The frequency could be varied between 0.01 and 26.5 GHz. The microwaves were transmitted to the resonator through a microwave cable; the reflected microwave power was measured as a function of frequency. Figure 1 shows part of a reflection spectrum for a stadium-shaped resonator. A spectrum as shown in Fig. 1 is registered within seconds. Typical times to measure a spectrum over the full frequency range including signal averaging and data transfer to an attached computer amount to about half an hour. In all resonators the top and the bottom face were parallel to each other with a distance of $d = 8$ mm. Thus for frequencies $\nu < \nu_{max} = c/2d = 18.74$ GHz the resonator can be considered as two dimensional. In all measurements described below only frequencies with $\nu < \nu_{max}$ were taken into account. In general, different

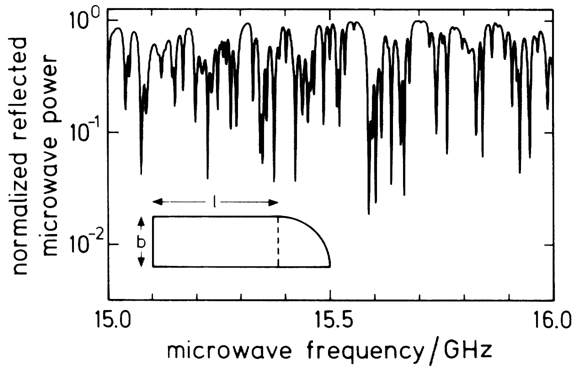

FIG. 1. Part of the spectrum of a stadium billiard ($b = 20$ cm, $l = 36$ cm). It was obtained by measuring the reflected microwave power as a function of the microwave frequency. Each resonance corresponds to an eigenfrequency of the billiard.

boundary conditions apply to quantum-mechanical and electromagnetic systems, respectively. Whereas in the quantum-mechanical case the wave function ψ must be zero at the boundary, in the electromagnetic case the tangential component E_{tang} of the electric field and the normal component B_{norm} of the magnetic induction must vanish at the boundary. For frequencies below ν_{max} only transverse magnetic modes are possible. Therefore in two-dimensional billiards the electromagnetic boundary conditions reduce to a single one, $E_z = 0$ on the boundary, where z is the direction perpendicular to the top and bottom faces of the resonator. Thus in the two-dimensional case quantum-mechanical and electromagnetic boundary conditions become identical, if one identifies ψ with E_z. Strictly speaking, E_{tang} is not exactly zero on the boundary because of the finite conductivity of the walls. This penetration of the electric field into the walls is the cause for the broadening of resonance lines and for the limited quality, as discussed above.

In the following we concentrate on one typical measurement for a Sinai billiard, rectangularly shaped with an excised quarter circle at one of the corners (see inset of Fig. 3). A total of 1002 eigenvalues was registered in the frequency range 1–18 GHz. It should be noted that in earlier calculations such as in Ref. 8 the total number of eigenvalues was only several hundred, even if the data of different billiards were combined. From the monotonic part $\rho_0(E)$ of the density of eigenvalues [see Eq. (2)] one would expect 1178 eigenvalues in the frequency range mentioned above. The loss of about 15% has two causes. First, the smallest distance which can be resolved because of the limited quality is about 3 MHz. This leads to a loss especially at small spacings, with increasing tendency at higher frequencies. Second, an eigenvalue may be overlooked in the reflection spectrum if the point where the microwave is coupled to the resonator is near a node of the eigenfunction (the depths of the

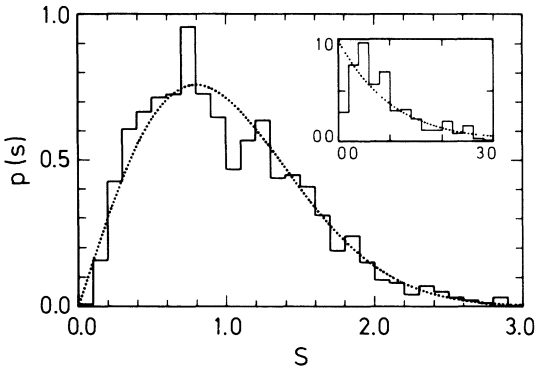

FIG. 2. Histogram for the distances s between successive eigenenergies for the Sinai billiard shown in the inset of Fig. 3. The dotted line corresponds to a Wigner distribution. Inset: The histogram of eigenfrequency distances for the rectangular billiard of the same size as the Sinai billiard. Now a Poisson distribution is observed. For the loss of eigenfrequencies at low distances see text.

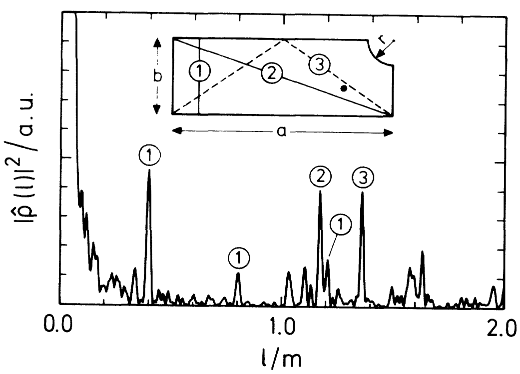

FIG. 3. Fourier transform of the spectrum of the Sinai billiard shown in the inset ($a = 56$ cm, $b = 20$ cm, $r = 7$ cm). The dot in the inset denotes the point where the microwaves were coupled to the resonator. Each resonance in the Fourier transform corresponds to a classical periodic orbit.

resonances in Fig. 1 are proportional to the square of the amplitude of the microwaves at the point of coupling).

Figure 2 shows the histogram of the distribution of distances of successive eigenenergies $s_n = E_n - E_{n-1}$ for a Sinai billiard. The "energy" is defined by $E = k^2$, and as in the quantum-mechanical analog, it is not the photon energy; $k = 2\pi\nu/c$ is the wave number of the microwave frequency. The abscissa of the figure is normalized to the mean distance $\bar{s} = 1$. The dotted line corresponds to a Wigner distribution

$$p(s) = \tfrac{1}{2}\pi s \exp(-\tfrac{1}{4}\pi s^2) . \quad (3)$$

The loss of eigenvalues at small distances mentioned above affects the lowest histogram value only. The inset of Fig. 2 shows the histogram of the distribution of eigenenergy distances for a rectangular billiard of the same size as the Sinai billiard. Now a Poisson distribution is observed, as it is expected for an integrable system. In this case only the 314 eigenfrequencies below 11 GHz were taken into account. For higher frequencies the spectrum showed more and more irregular behavior because of mechanical imperfections of the resonator.

Equation (1) shows that there is a close correspondence between eigenvalues and classical periodic orbits. For billiards the determination of periodic orbits is extremely simple, as only the laws of geometrical optics have to be obeyed. For the action of one special orbit one obtains $S_\gamma = \hbar k L_\gamma$, where L_γ is the length of the orbit. Therefore the Fourier transform of the density of eigenvalues, with k as the variable, i.e.,

$$\hat{\rho}(L) = \sum_n \exp(ik_n L) , \quad (4)$$

where the sum is over all eigenfrequencies, should show resonances at all L values corresponding to lengths of classical periodic orbits [see Eq. (1)]. Figure 3 shows $|\hat{\rho}(L)|^2$ for the same Sinai billiard as above. The billiard is shown in the inset of the figure, together with some elementary periodic orbits. A great number of resonances is seen. All of them could be attributed to classical periodic orbits. The resonances corresponding to the orbits shown in the inset are marked by numbers. For the "bouncing-ball" orbit 1 also higher-order resonances corresponding to multiple orbits are seen.

The fact that eigenfrequencies can be determined in very short times makes it possible to determine spectra for a large number of billiards with varying irregularity parameter. This is particularly simple for the stadium billiard. We varied the length l of the billiard shown in the inset of Fig. 1 between $l = 0$ and 5 cm. A small part of the spectra is shown in Fig. 4. First, one observes an overall decrease of eigenfrequencies with increasing

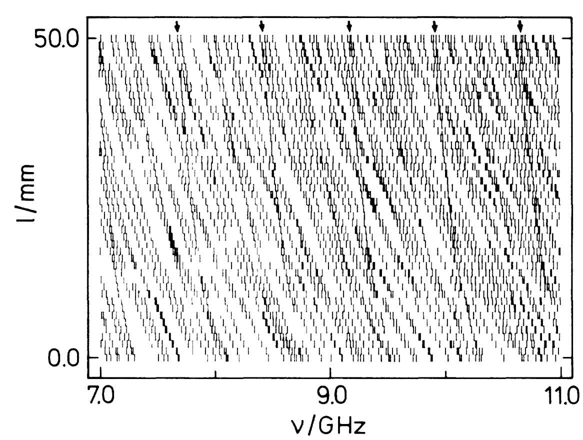

FIG. 4. Part of the spectra of different stadium billiards. The width of the billiard was fixed to $b = 20$ cm, the length l varied between 0 and 5 cm (see inset of Fig. 1). The periodic structures marked by arrows are due to eigenfrequencies associated with the "bouncing-ball" orbit.

length. This is a consequence of the fact that $\rho(E)$ is asymptotically proportional to the area of the billiard [see Eq. (2)]. The most conspicuous features of Fig. 4, however, are the regular patterns, marked by arrows at the upper margin of the figure. The distance of the frequency markers of $\Delta\nu = 0.75$ GHz corresponds to a length of $\Delta l = c/\Delta\nu = 40$ cm. This is exactly the length of the bouncing-ball orbit. The eigenfrequencies making up the periodic pattern therefore correspond to standing waves with node lines parallel to l. A somewhat more careful inspection of Fig. 4 shows that there are at least two further periodic patterns, with the same period length $\Delta\nu$, but with slopes differing from the slope of the pattern discussed above. The corresponding frequencies, too, must correspond to standing waves between the two long sides of the billiard, but with additional node lines parallel to the short side b. This example demonstrates that the correspondence between periodic orbits and eigenfrequencies can be seen even without performing a Fourier transform. Moreover, it is possible by direct inspection of the spectrum to associate part of the eigenfrequencies with specific periodic orbits. It is indispensible for this purpose, however, to plot the spectra for a large number of different l values together.

Though billiards represent only a small part of possible nonintegrable systems, they are especially suitable to study the close correspondence between spectra and classical periodic orbits. This work has shown that a great number of eigenvalues of billiards can be determined experimentally in very short times. A quantitative comparison with computations unfortunately is not possible to us as none of the publications mentioned in the beginning contains any information on computer time spent. In the experiments, contrary to the computations, the extension to three-dimensional systems is straightforward. Of course a price has to be paid. In the experiments it is unavoidable that part of the eigenvalues is overlooked, and one has to account for this fact if the statistical properties of the spectra are studied. If the correlation of the spectra with classical periodic orbits is considered, a loss of a few percent of the eigenvalues is probably of little relevance.

Future experiments will proceed in two directions: (i) continuation of the study of the dynamics of the eigenvalues under a change of the irregularity parameter of the billiard; (ii) systematic variation of the point where the microwaves are coupled to the resonator, by this also detailed information on wave functions and scars should be obtainable; (iii) measurement of the spectra of three-dimensional billiards. In this case the electromagnetic system is no longer equivalent to the quantum-mechanical one, the correspondence between eigenfrequencies and periodic orbits, however, still holds.

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