Electrodynamical Properties of Gapless Edge Excitations in the Fractional Quantum Hall States

X. G. Wen
School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540
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The electrodynamical properties of the edge excitations in the fractional quantum Hall (FQH) states are studied. We show how to experimentally measure the (optical) charges of the edge excitations. The interactions between the edge excitations are studied. We also discuss in detail the edge excitations of the $\nu=1\pm 1/n$ FQH states. Measuring the dynamical properties of the edge excitations is a practical way to probe the topological orders in the FQH states.

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Recently it was shown that the dynamical properties of the gapless edge excitations in the fractional quantum Hall (FQH) states are described by U(1) Kac-Moody algebras.[1] In general, the gapless excitations may contain many branches.[1,2] The excitations in each branch may carry a fractional charge. In this Letter we will show how to experimentally measure the fractional charge in each individual branch. We will study the response of the edge excitations to an external electric field.

The electrodynamical properties of the edge states in the integer quantum Hall (IQH) states have been extensively studied experimentally.[3] The experiments are sensitive enough to clearly observe the gapless excitations. The classical electrodynamical theory of the magnetoplasmon can be found in Ref. 4. Here we will emphasize on some new features appearing in the FQH states.

The response of the edge excitations to an external electric field is determined by the edge-current correlation function which has been studied in detail in Ref. 1. Let us first review some results in Ref. 1. Assume that a two-dimensional electron system demonstrates the FQH effect (or the IQH effect) in a background magnetic field $\bar{A}_i$ ($\bar{A}_0=0$). Because of the finite-energy gap, we can safely integrate out the electrons and obtain an effective Lagrangian

$$\mathcal{L}_{\text{eff}}=\frac{\nu e^2}{4\pi}\delta A_\mu \partial_\mu \delta A_\lambda \epsilon^{\mu\nu\lambda}+\frac{1}{4g_1^2}(\delta F_{0i})^2-\frac{1}{4g_2^2}(\delta F_{12})^2+\cdots, \tag{1}$$

where δA_μ is the perturbation around the constant magnetic field $\bar{A}_\mu$ and ν is the filling fraction. The coefficient of the Chern-Simons term $\delta A_\mu \partial_\nu \delta A_\lambda \epsilon^{\mu\nu\lambda}$ is given by the quantized Hall conductance.

On a compactified space, the action

$$S_{\text{bulk}}=\int d^3x\, \mathcal{L}_{\text{eff}}(\delta A_\mu) \tag{2}$$

is invariant under the gauge transformation

$$\delta A_\mu \to \delta A_\mu+\partial_\mu f(x). \tag{3}$$

However, on a space with boundary, say, a disk D, S_{bulk} is *not* gauge invariant:

$$S_{\text{bulk}}[\delta A_\mu+\partial_\mu f(x)]-S_{\text{bulk}}(\delta A_\mu)=\int dx_0\int_C dx^i \frac{\nu e^2}{4\pi}f(x)\delta F_{i0}=\int dx_0 d\sigma \frac{\nu e^2}{4\pi}f\,\delta F_{\sigma 0}, \tag{4}$$

where $C=\partial D$ is the boundary of the disk and σ parametrizes the boundary C. Because the microscopic theory is gauge invariant, (4) implies that S_{bulk} is not the complete action of the FQH states on the disk. Since the change in S_{bulk} is just a boundary term, the total gauge-invariant effective action may be obtained by including a boundary action associated with the edge excitations $S_{\text{tot}}=S_{\text{bulk}}+S_{\text{bd}}$. The effective action of the edge excitations has a form

$$S_{\text{bd}}=\int dt\, d\sigma\, dt'\, d\sigma'\, \tfrac{1}{2}\,\delta A_\alpha(t,\sigma)K^{\alpha\beta}(t-t',\sigma-\sigma')\times\delta A_\beta(t',\sigma'), \tag{5}$$

where $K^{\alpha\beta}$ is the current correlation function of the edge excitations. Using the gauge invariance of the total action S_{tot}, we can determine the possible form of the edge-current correlation function $K^{\alpha\beta}$. First, one can show that the edge excitations must be gapless as a consequence of the gauge invariance of S_{tot}. At low energies, the gapless edge excitations, in general, may consist of many branches. The excitations in each branch have a common velocity v_I. In this case the gauge invariance of S_{tot} requires the edge-current correlation function to have a form (up to a polynomial in ω and k)[1]

$$K^{00}=-\sum_I \frac{k}{\omega-v_I k+i\delta_I}\frac{q_I^2 v_I}{2\pi|v_I|},$$
$$K^{\sigma 0}=K^{0\sigma}=-\tfrac{1}{2}\sum_I \frac{\omega+v_I k}{\omega-v_I k+i\delta_I}\frac{q_I^2 v_I}{2\pi|v_I|}, \tag{6}$$
$$K^{\sigma\sigma}=-\sum_I \frac{v_I\omega}{\omega-v_I k+i\delta_I}\frac{q_I^2 v_I}{2\pi|v_I|},$$

where the constant q_I satisfies the sum rule

$$\sum_I \frac{v_I}{|v_I|}q_I^2=\nu e^2. \tag{7}$$

From the current correlation function $K^{\alpha\beta}$ we can further show that (using the locality properties of the theory) the edge currents satisfy the U(1) Kac-Moody (KM) algebra:

$$[\rho_{Ik},\rho_{Jk'}]=\mathrm{sgn}(v_I)\frac{q_I^2}{2\pi}\delta_{k+k'}\delta_{I,J}\,,$$
$$[H,\rho_{Ik}]=v_I k\rho_{Ik}\,, \tag{8}$$

where ρ_I $(=j_I^0)$ and v_I are the charge density and the velocity of the edge excitations in the Ith branch. The edge current associated with the Ith branch is given by $j_I=j_I^\sigma=v_I\rho_I$. The KM algebra (8) contains several independent copies. Each copy corresponds to one branch of the edge excitations. From the experience of the Tomonaga model,[5] we see that the algebra (8) completely determines the dynamics of the edge excitations.

We know that the edge excitations of the IQH states are described by the Fermi-liquid theory of the charge e electrons.[6] In order to compare the edge excitations of the FQH states to the IQH states, we would like to point out that the edge excitations in the FQH states can also be regarded as a Fermi liquid, however, in a restricted sense (see the remark at the end of the paper). Such a Fermi liquid contains several branches of fermions.[1] The charges and the Fermi velocities of the fermions are given by q_I and v_I, respectively. One can check that, in such a Fermi-liquid theory, the charge densities satisfy the algebra (8) and the currents reproduce the correlation function $K^{\alpha\beta}$. We will call q_I the optical charges (or simply the charges) of the edge excitations. We see that, in contrast to the IQH states, the edge excitations of the FQH states are related to a Fermi-liquid theory with nonintegral charges q_I.

We would like to stress that the fermions in the above Fermi-liquid theory do not correspond to the electrons. They are just effective fields describing the collective excitations on the edge. They correspond to the Tomonaga bosons in the Tomonaga model. This is possible because bosons and fermions are equivalent in one dimension.

Let us study the response of the edge excitations to a uniform rotating electric field. We will assume that the FQH state has a disk geometry (see Fig. 1). In this case δA_μ are given by

$$\delta A_0(t,\sigma)=\phi(t,\sigma)=R\cos\left(\omega t-\frac{\sigma}{R}\right)E\,,$$
$$\delta A_\sigma(t,\sigma)=0\,, \tag{9}$$

where R is the radius of the disk and E is the strength of the electric field. A positive (negative) frequency ω corresponds to counterclockwise- (clockwise-) rotating electric field. The induced charge and the current density of the edge states are given by

$$j^\mu(\omega,k)=K^{\mu\nu}(\omega,k)\delta A_\nu(\omega,k)\,. \tag{10}$$

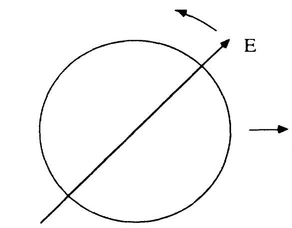

FIG. 1. The disklike FQH state and the rotating electric field.

Using (10) we may calculate the dipole moment induced by the uniform electric field

$$P_x(t)=\int d\sigma\, j^0(t,\sigma)R\cos\left(\frac{\sigma}{R}\right)=\chi E\cos(\omega t)\,. \tag{11}$$

The susceptibility χ is given by

$$\chi=-\frac{\alpha}{2}R^3\sum_I\frac{v_I}{\omega R-v_I+i\delta_I R}\frac{c}{|v_I|}\frac{q_I^2}{e^2}\,, \tag{12}$$

where $\alpha=e^2/\hbar c\simeq 1/137$ is the fine-structure constant and c is the speed of light. The imaginary part of χ has many δ-function peaks at the resonance frequencies $f_I=\omega_I/2\pi=v_I/2\pi R$. The strength of the δ-function peaks

$$\int_{\omega_I-\delta\omega}^{\omega_I+\delta\omega}\mathrm{Im}\chi(\omega)d\omega=\frac{\pi}{2}\alpha R^2 c\frac{q_I^2}{e^2} \tag{13}$$

measures the optical charges carried by the edge excitations. The poles at positive or negative frequencies correspond to the resonances with counterclockwise- or clockwise-rotating electric fields.

In realistic samples, δ_I in (6) are finite. The dc resistivity ρ_{edge} along the edge is given by

$$\frac{1}{\rho_{\mathrm{edge}}}=\frac{1}{\omega}\mathrm{Im}K^{\sigma\sigma}(\omega,k)\big|_{\omega,k\to 0}=\frac{e^2}{h}\sum_I\frac{|v_I|q_I^2}{\delta_I e^2}\,. \tag{14}$$

Note ρ_{edge} is a resistivity in one dimension $\rho_{\mathrm{edge}}=R_{\mathrm{edge}}/L$. Now the δ-function peaks in $\mathrm{Im}\chi(f)$ have finite width. The width at half strength is given by

$$\Delta f_I=\delta_I/\pi\,. \tag{15}$$

From (14) and (15) we see that the widths of the resonances and the edge resistivity ρ_{edge} are related:

$$\rho_{\mathrm{edge}}=\frac{h}{e^2}\left(\sum\frac{2\pi R f_I}{\pi\Delta f_I}\frac{q_I^2}{e^2}\right)^{-1}. \tag{16}$$

Let us study the interactions between edge excitations. The Hamiltonian of the edge excitations satisfies (8) and is given by

$$H=\sum_I\frac{\pi}{q_I^2}|v_I|\rho_I^2\,. \tag{17}$$

The coupling between the charge density and the external electric potential is

$$\sum_I \rho_I(\sigma)\phi(\sigma)\,. \tag{18}$$

The Hamiltonian (17) is valid only for a specific electron interaction and edge configuration. If we modify the electron interaction and edge configuration, the Hamiltonian (17) is expected to receive correctons:

$$\tilde{H}=H+\delta H,\quad \delta H=\sum_{I,J} V_{IJ}\rho_I\rho_J\,. \tag{19}$$

The modified Hamiltonian $\tilde{H}$ can be diagonalized by introducing $\tilde{\rho}_I$:

$$\frac{1}{\tilde{q}_I}\tilde{\rho}_I=\sum_J U_{IJ}\frac{1}{q_J}\rho_J\,, \tag{20}$$

where U_{IJ} satisfies

$$U_{II'}U_{JJ'}\eta_{I'J'}=\eta_{IJ},\quad \eta_{IJ}=0\,|_{I\neq J},\quad \eta_{II}=\mathrm{sgn}(v_I)\,. \tag{21}$$

In terms of $\tilde{\rho}_I$, $\tilde{H}$ has the form

$$\tilde{H}=\sum_I \frac{\pi}{\tilde{q}_I^2}|\tilde{v}_I|\tilde{\rho}_I^2\,. \tag{22}$$

One can show that such a matrix U_{IJ} always exists as long as $\tilde{H}$ is a positive-definite function of ρ_I. Because of (21) the algebra of $\tilde{\rho}_I$ is still diagonal:

$$[\tilde{\rho}_{Ik},\tilde{\rho}_{Jk'}]=\mathrm{sgn}(\tilde{v}_I)\frac{\tilde{q}_I^2}{2\pi}\delta_{k+k'}\delta_{I,J}\,. \tag{23}$$

From (22) and (23) we see that each $\tilde{\rho}_I$ describes a branch of edge excitations with a new velocity $\tilde{v}_I$. If we choose

$$\tilde{q}_I=\sum_J U_{IJ}q_J\,, \tag{24}$$

we find the coupling between $\tilde{\rho}_I$ and ϕ to be

$$\sum_I \tilde{\rho}_I(\sigma)\phi(\sigma)\,. \tag{25}$$

Thus $\tilde{\rho}_I$ can be still regarded as the charge density and $\tilde{q}_I$ are the new optical charges [see (23)]. We see that after including the interaction V_{IJ}, the original edge excitations with the optical charge q_I and the velocity v_I are mixed with each other and form new branches. The edge excitations in the new branches have different optical charges and velocities which are given by $\tilde{q}_I$ and $\tilde{v}_I$.

Let us consider the IQH states in more detail. Assume all branches have the same velocity $v_I=v$ and the interaction V_{IJ} is only between neighboring branches:

$$V_{IJ}=V\delta_{I,J+1}+V\delta_{I,J-1}\,. \tag{26}$$

We find the optical charges of the new branches are given by

$$\nu=2:\ (\tilde{q}_1,\tilde{q}_2)=(\sqrt{2},0)\,,$$

$$\nu=3:\ (\tilde{q}_1,\tilde{q}_2,\tilde{q}_3)=(1.71,0,0.29)\,,$$

$$\nu=4:\ (\tilde{q}_1,\tilde{q}_2,\tilde{q}_3,\tilde{q}_4)=(1.942,0,0.48,0)\,.$$

We see that for the $\nu=2$ IQH state only one branch of the edge magnetoplasmons is observable. For $\nu=3$ and $\nu=4$ we can only observe two branches of edge magnetoplasmons. The second branch is much weaker than the first one because $(\tilde{q}_3/\tilde{q}_1)^2$ is equal to $\frac{1}{32}$ for $\nu=3$ and $\frac{1}{18}$ for $\nu=4$.

Haldane[7] has argued that for the FQH states with simple filling fractions $\nu=1/n$ the specific heat of the edge excitations is equal to the specific heat arising from a single Fermi point. This implies that there is only one branch of the edge excitations. According to (7) the fermions in the corresponding Fermi-liquid theory must carry an *irrational* (optical) charge $q=1/\sqrt{n}$. The velocity of the edge excitations is roughly given by $v\simeq cE/B$, where E is the electric field normal to the boundary and B is the magnetic field.

Using the above results we would like to argue that the FQH states with filling fractions $\nu=1-1/n$ have two branches of edge excitations. To understand this result let us first consider a special edge configuration. The edge potential is arranged such that the filling fraction first changes from $1-1/n$ to 1, and then from 1 to 0 (see Fig. 2). The edge excitations on the boundary between the $\nu=1-1/n$ and the $\nu=1$ FQH states are identical to the edge excitations on the boundary of the $\nu=1/n$ FQH state. They carry an optical charge $q_2=1/\sqrt{n}$. The edge excitations on the outer boundary are just the edge excitations of the $\nu=1$ IQH state[6] with an optical charge $q_1=1$. Because the electric fields on the two boundaries point to opposite directions, the edge excitations on the two boundaries have opposite velocities. As we deform the edge potential, the two edge branches may come close together and start to interact with each other. The optical charges and the velocities may change due to the interaction. However, the number of branches and the signs of the velocities remain unchanged, at least when the interaction is not too strong.

Similarly we can argue that there are two branches of edge excitations in the FQH states with filling fractions

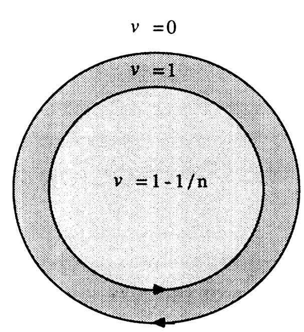

FIG. 2. A special edge configuration of the $\nu=1-1/n$ FQH states. The edge excitations on the outer boundary carry an optical charge $q_1=1$. Those on the inner boundary carry an optical charge $q_2=1/\sqrt{n}$. The excitations on the two boundaries move in opposite directions.

$\nu=1+1/n$. The optical charges are given by $q_1=1$ and $q_2=1/\sqrt{n}$ if the interaction is ignored. But in this case the excitations in the two branches move in the *same* direction.

Before ending this paper we would like to make the following remark. As have been pointed out in Ref. 1, the Fermi-liquid description of the edge excitations are only valid in the charge-zero sector when q_I are not integers. We can only prove that the charge-zero sector of the Fermi-liquid theory is equivalent to the charge-zero sector of the edge excitations (which is described by the KM algebra). The properties of the charged excited states cannot be determined solely from the Kac-Moody algebra. The fact that the charge-zero excited states are described by a charge q_I Fermi-liquid theory does not imply that the total charge of a charged excited state is a multiple of q_I. In our case the charged excited states are obtained by adding electrons to the edge. Therefore the total charges of charged excited states are quantized as integers. The optical charge q_I is measured by the current correlation function which involves only the charge-zero section. The mismatch between the optical charge and the total charges of charged excited states implies that the charged excited states are not described by the Fermi-liquid theory.[8] Therefore is is not surprising to see that the charge-zero sector of the edge excitations is described by a Fermi-liquid theory with an irrational charge. The fermions in the Fermi-liquid theory are just effective fields describing the collective edge excitations.

However, it is still safe and helpful to use the Fermi-liquid description of the edge states if we are only concerned about the processes that conserve the total charge. For example, the Fermi-liquid theory gives the correct value of specific heat. It also gives a correct description of the response to external electromagnetic fields.

In this paper we study the electrodynamical properties of the edge excitations in the FQH states. We show how to measure the optical charges of the edge excitations. Measuring the dynamical properties of the edge excitations is a practical way to probe the hierarchy structures, or more precisely, the topological orders[9] in the FQH states. The edge excitations provide us with a window through which we can look into the internal structures of the FQH states.

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[1]X. G. Wen, Institute for Theoretical Physics Report No. NSF-ITP-89-157 (to be published).

[2]C. W. J. Beenakker, Phys. Rev. Lett. **64**, 216 (1990). Some results in Wen (Ref. 1) and in A. H. MacDonald [Phys. Rev. Lett. **64**, 220 (1990)] appear to be different.

[3]S. J. Allen, Jr., H. L. Störmer, and J. C. M. Hwang, Phys. Rev. B **28**, 4875 (1983); D. B. Mast, A. J. Dahm, and A. L. Fetter, Phys. Rev. Lett. **54**, 1706 (1985); D. C. Glattli *et al.*, Phys. Rev. Lett. **54**, 1710 (1985); S. A. Govorkov *et al.*, Pis'ma Zh. Eksp. Teor. Fiz. **45**, 252 (1987) [JETP Lett. **45**, 317 (1987)]; Zh. Eksp. Teor. Fiz. **94**, 226 (1988) [Sov. Phys. JETP **67**, 342 (1988)]; I. E. Batove *et al.* (to be published).

[4]Mast, Dahm, and Fetter, Ref. 3; J. W. Wu, P. Hawrylak, and J. J. Quinn, Phys. Rev. Lett. **55**, 879 (1985); V. A. Volkov and S. A. Mikhailov, Pis'ma Zh. Eksp. Teor. Fiz. **42**, 450 (1985) [JETP Lett. **42**, 556 (1985)].

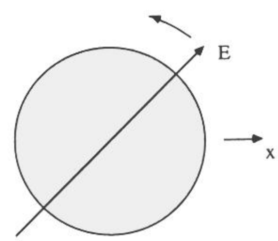
[5]G. D. Mahan, *Many-Particle Physics* (Plenum, New York, 1981), p. 311.

[6]B. I. Halperin, Phys. Rev. B **25**, 2185 (1982).

[7]F. D. M. Haldane (private communication).

[8]X. G. Wen, Institute for Advanced Study, Princeton University Report No. IASSNS-HEP-90/14 (to be published).

[9]X. G. Wen, Phys. Rev. B **40**, 7387 (1989); (to be published); X. G. Wen and Q. Niu, Phys. Rev. B (to be published).



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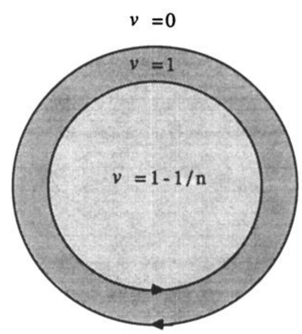

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