Edge States in the Fractional-Quantum-Hall-Effect Regime

A. H. MacDonald
Department of Physics, Indiana University, Bloomington, Indiana 47405
(Received 16 October 1989)

A finite two-dimensional electron gas with N full Landau levels has N branches of edge states which cross the Fermi level. In this Letter we show that in the fractional-quantum-Hall-effect regime there can be many branches of edge states for a single partly filled Landau level. The ith branch can be associated with a fractional charge, ef_i, and $\sum_i f_i$ equals the Landau-level filling factor. The set of edge-state charges at a particular filling factor directly reflects the hierarchical structure of the incompressible ground state which occurs at that filling factor.

PACS numbers: 73.40.Kp, 67.50.−b

In the thermodynamic limit a noninteracting two-dimensional electron gas (2DEG) has an incompressible ground state whenever an integral number of Landau levels is full; i.e., the chemical potential jumps from $\hbar\omega_c(N+\frac{1}{2})$ to $\hbar\omega_c(N+\frac{3}{2})$ after the Nth Landau level is filled. It follows that for a finite system all single-particle states which occur at energies in these gaps must be localized at the edge of the system. In fact it is easy to show that there is one branch of edge states for each occupied Landau level and that the states in each branch may be labeled by the number of flux quanta ($\Phi_0=hc/e$) that their orbits enclose.[1]

The importance of these edge states in understanding the quantum Hall effect (QHE) has been appreciated[2] since shortly after the effect's discovery.[3] More recently[4] the edge-state picture has been placed in the framework of Landauer resistance formulas,[5] allowing for a unified description of the quantum Hall effect and other quantum transport phenomena.[6] In this Letter we generalize the edge-state picture to the fractional-quantum-Hall-effect (FQHE) regime.[7] We predict the existence of a set of edge-state branches. The number of edge-state branches which occur at a particular filling factor and the set of branch charges are determined by the hierarchical[8] structure of the incompressible ground state which occurs at that filling factor.

The starting point for the construction of the hierarchy is the set of incompressible states discovered by Laughlin,[9]

$$\Phi_m[z]=\prod_{i<j}(z_i-z_j)^m\prod_k \exp(-|z_k|^2/4l^2)\,. \qquad (1)$$

For $m=1$ this state is the full-Laudau-level state in which all single-particle states,

$$\varphi_k(z)=\frac{z^k}{(2\pi 2^k k!l)^{1/2}}\exp\left(-\frac{|z|^2}{4l^2}\right), \qquad (2)$$

from $k=0$ to $k=N-1$ are occupied. [In Eqs. (1) and (2) $z=x-iy$ and $2\pi l^2B=\Phi_0$ so that $\varphi_{k+1}(z)$ encloses one more quantum of flux than $\varphi_k(z)$.] For a finite system[10] the energies of these single-particle states increase with m at the edge of the system, as illustrated schematically in Fig. 1(a). For each added electron the flux enclosed by the system increases by Φ_0. For $m\neq 1$ it is easily verified by comparing Eqs. (1) and (2) that the flux enclosed by the system increases by $m\Phi_0$ for each electron added to the system. The states which can be constructed with a smaller increase in area must place electrons at a lower relative angular momentum and therefore lie above the bulk excitation gap[11] [see Fig. 1(b)]. The edge states which exist in the gap are therefore quantized as if[9] their flux quantum were $\Phi_0^*=hc/e^*$ with $e^*=ef$ and $f=1/m$. We will assign a charge to each branch of edge states according to the rate of increase of enclosed flux as particles are added in that

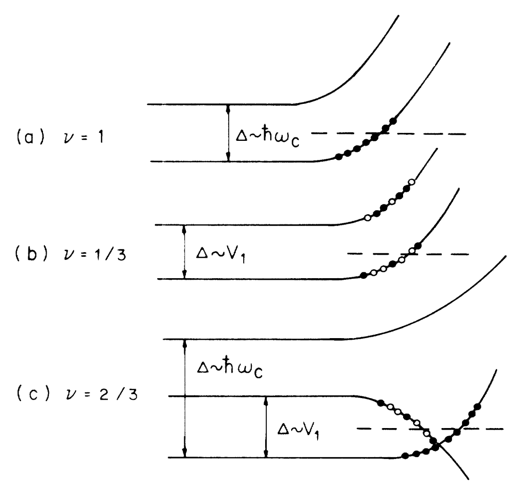

FIG. 1. Schematic comparison of edge-state branches in (a) integer and (b),(c) fractional regimes. When the chemical potential (dashed line) lies in the gap electrons are added or taken away at the edge of the system when the chemical potential is changed. (b) At $\nu=1/m$ the area of the system must enclose m additional flux quanta for each electron added to the system to avoid placing electrons at relative angular momentum $m-2$ and raising the chemical potential above the gap. (c) For $\nu=1-1/m$ an electron can be added by taking the particle-hole conjugate with respect to a system with an additional electron or by taking the particle-hole conjugate of a state with one fewer electron.

branch.

An essential element of the hierarchy construction, and the one which leads to a proliferation of edge-state branches, is the invocation of particle-hole symmetry.[12] Given an incompressible state, Φ, which occurs at $\nu < \frac{1}{2}$ we can conclude that the particle-hole conjugate of Φ is an incompressible ground state at filling factor $\nu' = 1 - \nu$. For a finite system a new branch of edge states is introduced by this process, corresponding to the edge states of the full Landau level with respect to which the particle-hole conjugation is taken. In addition, the edge-state branches which already existed in Φ change from having electronlike character to having holelike character, or from having holelike character to having electronlike character. For edge branches of holelike character the number of flux quantum enclosed by the orbit *decreases* as the chemical potential is increased [see Fig. 1(c)]. Thus, for example, at $\nu = \frac{2}{3}$ there are two edge-state branches, one of which has charge 1 while the other has charge $-\frac{1}{3}$.

A prescription for generating the full hierarchy may be based on the process of particle-hole conjugation.[13,14] We can always express the particle-hole conjugate of an incompressible state Φ in the form

$$\Phi_1^C[z] = Q[z]\Phi_1[z] . \tag{3}$$

($Q[z]$ is defined by this equation.) Since $\Phi_1[z]$ is the full-Landau-level wave function, $Q[z]$ is the function which creates the incompressible state Φ in the holes of the full Landau level. The wave function where the state represented by Φ is created in the *quasiholes* of $\Phi_m[z]$ is then given by

$$\Phi_m^C[z] = Q[z]\Phi_m[z] , \quad m = 1,3,\ldots, \tag{4}$$

and occurs at filling factor

$$\nu_m^C = (1-\nu)/[m(1-\nu)+\nu] . \tag{5}$$

(Here ν is the filling factor at which $\Phi[z]$ occurs.) Comparing Eqs. (1) and (2) we see that replacing $\Phi_1[z]$ on the right-hand side of Eq. (3) with $\Phi_m[z]$ on the right-hand side of Eq. (4) expands the area associated with each edge state by the factor[15] $\nu_1^C/\nu_m^C = m(1-\nu)+\nu$. Thus the flux quantum for the edge states is increased, and the characteristic charge decreased, by the same factor. The wave function where the state represented by $\Phi[z]$ is created in the *quasiparticles* of $\Phi_m[z]$ is then given by[13,14]

$$\tilde{\Phi}_m^C[z] = Q^+[z]\Phi_m[z] , \quad m = 3,5,7,\ldots, \tag{6}$$

where $Q^+[z]$ is obtained from $Q[z]$ by replacing[16] z_i^k by $(2\partial_i)^k$ and thus acts to contract the system rather than expand it. This incompressible state occurs at filling factor

$$\tilde{\nu}_m^C = (1-\nu)/[m(1-\nu)-\nu] . \tag{7}$$

Since the edge-state branches which occur in $\Phi[z]$ are reflected entirely in $Q[z]$, these branches will now have the same sign as in $\Phi[z]$ but will have their charges scaled by the area expansion factor $\nu_1^C/\tilde{\nu}_m^C$. However, occupying a new state in the full Landau level before particle-hole conjugation changes both $\Phi_1[z]$ and $Q[z]$. By calculating the change in the degree of $\tilde{\Phi}_m^C[z]$ when one electron is added to the full Landau level we may conclude that for this case the charge of the added branch is $(1-2\nu)\tilde{\nu}_m^C/\nu_1^C(1-\nu)$.

The conclusions from the preceding paragraphs can be summarized by the following rules. The incompressible states at the first level of the hierarchy are the Laughlin states, $\Phi_m[z]$. These states have a single branch of edge states with charge $e_1 = ef_1 = e/m$. For any hierarchy state which occurs at filling factor $\nu < \frac{1}{2}$, a series of holelike daughter states occurs at the next level of the hierarchy at filling factors $\nu' = \nu_m^C$ [Eq. (5)]. If Φ has M branches of edge states, with charge ef_i in branch i, then the daughter state has M branches of edge states with charges given by

$$f_i' = -f_i \nu_m^C/\nu_1^C , \quad i = 1,\ldots,M , \tag{8}$$

and an additional branch with charge

$$f_{M+1}' = \nu_m^C/\nu_1^C . \tag{9}$$

Similarly the particlelike daughter states occur at filling factors $\nu' = \tilde{\nu}_m^C$ and have charges given by

$$f_i' = f_i \tilde{\nu}_m^C/\nu_1^C , \quad i = 1,\ldots,M , \tag{10}$$

and

$$f_{M+1}' = [(1-2\nu)/(1-\nu)]\tilde{\nu}_m^C/\nu_1^C . \tag{11}$$

Note that a state which occurs at level M of the hierarchy has M branches of edge states. Also

$$\sum_{i=1}^{M+1} f_i' = \nu' \tag{12}$$

for both electronlike and holelike daughters. Equation (12) can be proved by induction starting from the Laughlin states and using Eqs. (8)–(11). We see below that Eq. (12) is required by the occurrence of the fractional quantum Hall effect. The edge-state branch charges for some states occurring at low levels of the hierarchy are listed in Table I.

States in the gap at level M of the hierarchy are completely specified by the single-particle edge-state occupancies of the M full-Landau-level wave functions which enter into their hierarchical construction. The parametrization of the relevant many-body states in terms of single-particle quantum numbers has some similarity to the Landau Fermi-liquid phenomenology for strongly interacting metals and allows us to use single-particle concepts to understand low-energy properties. As we discuss below, many of these properties depend only on the quasiparticle charges listed in Table I. The low-lying ex-

TABLE I. Fractional charges of edge-state branches for some incompressible states occurring at low levels of the hierarchy.

M	ν	e_1	e_2	e_3	e_4
2	$\frac{2}{3}$	1	$-\frac{1}{3}$		
2	$\frac{2}{7}$	$\frac{3}{7}$	$-\frac{1}{7}$		
2	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$		
3	$\frac{5}{7}$	$\frac{1}{7}$	$-\frac{3}{7}$	1	
3	$\frac{5}{17}$	$\frac{1}{17}$	$-\frac{3}{17}$	$\frac{7}{17}$	
3	$\frac{5}{13}$	$-\frac{1}{13}$	$\frac{3}{13}$	$\frac{3}{13}$	
3	$\frac{3}{5}$	$-\frac{1}{5}$	$-\frac{1}{5}$	1	
3	$\frac{3}{11}$	$-\frac{1}{11}$	$-\frac{1}{11}$	$\frac{5}{11}$	
3	$\frac{3}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	
4	$\frac{4}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

citations at level M of the hierarchy are much like those of a system of noninteracting electrons with M full Landau levels except that the charges may be fractional and may be negative. As emphasized elsewhere[17] and discussed explicitly below, the occurrence of the fractional quantum Hall effect at some filling depends only on the existence of an incompressibility at that filling factor and is quite independent of the microscopic origin of that incompressibility. On the other hand, the charges listed in Table I directly reflect the hierarchical nature of fractional Hall states.

As in the integer case[1,4,17] we assert that on the fractional Hall plateaus local equilibria are established on opposite edges of a system which carries a net current. The Hall conductance is then e times the rate of change of the diamagnetic edge current with the chemical potential *in equilibrium*. This may be related to thermodynamic properties of the system since, when the chemical potential lies in the gap,

$$\left.\frac{\partial M}{\partial \mu}\right|_B = \left\langle -\frac{\partial \hat{H}}{\partial B}\right\rangle = \frac{1}{2c}\int d^2x \left[\mathbf{x}\times\frac{\partial\langle\hat{\mathbf{j}}(\mathbf{x})\rangle}{\partial\mu}\right]_z = \frac{A}{c}\frac{\partial I_e}{\partial \mu}. \quad (13)$$

The last form for Eq. (13) follows from the continuity equation and from the fact that $\partial\langle\mathbf{j}(x)\rangle/\partial\mu$ can be nonzero only at the edge of a gapful system. We evaluate $\partial M/\partial\mu|_B$ from the expression ($T=0$)

$$\left.\frac{\partial M}{\partial \mu}\right|_B = -\frac{\partial^2}{\partial\mu\partial B}[E-\mu N] = -\frac{\partial^2}{\partial\mu\partial B}\left[\sum_{i=1}^{M}\frac{Bf_i}{\Phi_0}\int_0^\infty dA[E_i(A)-\mu]\theta(\mu-E_i(A))\right]. \quad (14)$$

In Eq. (14) we have gone to the continuum limit and labeled states by the areas they enclose. The quasiparticle energies, $\{E_i(A)\}$, are defined to be the energy *per electron* to add a quasiparticle in branch i and we have used the fact that for branch i Bf_i/Φ_0 electrons are added to the system per unit area increase. Combining Eqs. (12), (13), and (14) we have the expected result for the Hall conductance,

$$G_H = e\frac{\partial I_e}{\partial\mu} = \frac{ec}{A}\left.\frac{\partial M}{\partial\mu}\right|_B = \frac{e^2}{h}\sum_{i=1}^{M} f_i = \frac{e^2\nu}{h}. \quad (15)$$

Since $\partial M/\partial\mu|_B = \partial N/\partial B|_\mu$ it follows from Eq. (14) that the fractional Hall effect will occur whenever a gap exists at a fractional filling factor and hence that Eq. (12) must be satisfied by any theory of the fractional edge states regardless of its microscopic origin.

Since the fractional Hall effect only depends on the total of all branch charges it is of interest to be able to describe more general situations where the various edge-state branches are not always in local equilibrium. We do so by generalizing the Büttiker multichannel multiprobe resistance formula[4,5] to the fractional regime. The current out of the αth probe is given by

$$I_\alpha = \frac{e}{h}\sum_{\beta\neq\alpha} T_{\alpha,\beta}(\mu_\alpha - \mu_\beta), \quad (16)$$

where we use greek indices as probe labels,

$$T_{\alpha,\beta} = \sum_{i=1}^{M_\alpha}\sum_{j=1}^{M_\beta} f_j P_{\alpha,\beta}^{i,j}, \quad (17)$$

and $P_{\alpha,\beta}^{i,j}$ is the transmission probability from branch j in lead β to branch i in lead α. Equation (16) expresses the total current in lead α as the total outgoing current less the incoming current from all other contacts. Thus $eT_{\alpha,\beta}/h$ is the increase in the current arriving at probe α per unit chemical-potential change at probe β. Equation (17) then follows from Eq. (15). Equation (17) provides a basis for discussing localization phenomena in the fractional regime and for analysis of experiments in which gates are used to establish contiguous regions which are fractionally quantized at different densities.[18] The latter experiments can potentially provide unique information about the incompressible ground states as we illustrate with the situation depicted in Fig. 2, where current is injected from a region exhibiting the $\nu=\frac{2}{3}$ FQHE through a barrier exhibiting the $\nu=1$ integer QHE and collected in a $\nu=\frac{2}{3}$ FQHE region. The charge 1 edge branch exists in both regions, while the holelike charge $-\frac{1}{3}$ branch sees the electrostatic potential which attracts electrons to the higher-density region as repulsive. This suggests that it may be possible to realize a situation where the hole branch would be reflected. Application of Eqs. (16) and (17) would then lead to a negative resistance as illustrated in Fig. 2. Confirmation of this prediction would constitute the first direct experimental demonstration of the hierarchical nature of the FQHE ground states. We emphasize that the situation depicted in Fig. 2 will be realized only if the two edge branches

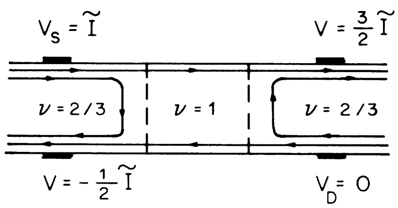

FIG. 2. Transmission probabilities for injection from a $\nu=\frac{2}{3}$ FQHE region toward a $\nu=1$ integer QHE region. Carriers in the holelike edge branch are repelled by the potential under the gate while carriers in the charge 1 edge branch are transmitted. Disorder will prevent the reflection from taking place. $\tilde{I}=-hI/e^2$. Note that the chemical potential increases in the direction of net electron current flow.

are very weakly mixed by potential fluctuations near the barrier region and probably requires that the two edge-state branches be spatially separated by several magnetic lengths. Since the separation can be at most of the order of the distance over which the confining potential changes by the fractional excitation gap, the confining electric field at the edge cannot be more than a few times 10^2 V/cm at 10 T. (The analogous requirement for the integer case is that the electric field be less than about 10^4 V/cm.) When this condition is not satisfied the experiments will be sensitive only to the total edge charge, i.e., to the filling factor in each region. As far as we are aware all experiments to date[19,20] are in this regime.

The author acknowledges discussion with many colleagues and would especially like to than M. Büttiker, G. S. Canright, A. M. Chang, J. P. Eisenstein, S. M. Girvin, R. J. Haug, M. D. Johnson, K. von Klitzing, and P. Středa. This work was supported by the National Science Foundation under Grant No. DMR-8802383.

[1]Stuart A. Trugman, Phys. Rev. B **27**, 7539 (1983); J. K. Jain and S. A. Kivelson, Phys. Rev. A **36**, 3476 (1987); Phys. Rev. B **37**, 4111 (1988); A. H. MacDonald, in *Recent Progress in Many-Body Theories*, edited by E. Pajanne and A. Kallio (Plenum, New York, 1988).

[2]R. B. Laughlin, Phys. Rev. B **23**, 5632 (1981); B. I. Halperin, Phys. Rev. B **25**, 2185 (1982); A. H. MacDonald and P. Středa, Phys. Rev. B **29**, 1616 (1984).

[3]K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).

[4]P. Středa, J. Kucera, and A. H. MacDonald, Phys. Rev. Lett. **59**, 1973 (1987); J. K. Jain and S. A. Kivelson, Phys. Rev. B **37**, 4276 (1988); Phys. Rev. Lett. **60**, 1542 (1988); M. Büttiker, Phys. Rev. B **38**, 9375 (1988).

[5]R. Landauer, Philos. Mag. **21**, 863 (1970); M. Büttiker, Phys. Rev. Lett. **57**, 1761 (1986).

[6]M. Büttiker, in Proceedings of the Eighth International Conference on Electronic Properties of Two-Dimensional Systems, Grenoble, 1989 [Surf. Sci. (to be published)], and work quoted therein.

[7]D. C. Tsui, H. L. Störmer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).

[8]F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983); B. I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984).

[9]R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).

[10]For simplicity we assume a circularly symmetric disk geometry and an edge potential that is smooth on the scale of l.

[11]See F. D. M. Haldane, in *The Quantum Hall Effect*, edited by R. E. Prange and S. M. Girvin (Springer-Verlag, New York, 1987).

[12]S. M. Girvin, Phys. Rev. B **29**, 5617 (1984).

[13]R. B. Laughlin, Surf. Sci. **141**, 11 (1984).

[14]A. H. MacDonald and D. B. Murray, Phys. Rev. B **32**, 2707 (1985); A. H. MacDonald, G. C. Aers, and M. W. C. Dharma-wardana, Phys. Rev. B **31**, 5529 (1985).

[15]See R. B. Laughlin, in *The Quantum Hall Effect*, edited by R. E. Prange and S. M. Girvin (Springer-Verlag, New York, 1987).

[16]S. M. Girvin and Terrance Jach, Phys. Rev. B **29**, 5617 (1984).

[17]*The Quantum Hall Effect: A Perspective*, edited by A. H. MacDonald (Riedel, Dordrecht, 1990).

[18]D. A. Syphers and P. J. Stiles, Phys. Rev. B **32**, 6620 (1985); R. J. Haug, A. H. MacDonald, P. Středa, and K. von Klitzing, Phys. Rev. Lett. **61**, 2797 (1988); S. Washburn, A. B. Fowler, H. Schmid, and O. Kem, Phys. Rev. Lett. **61**, 2801 (1988); S. Komijama, H. Hira, S. Sasoand, and S. Yiyamizu, Phys. Rev. B **40**, 5176 (1989).

[19]A. M. Chang, Solid State Commun. (to be published); R. J. Haug (private communication).

[20]Typical electric fields at the edge of a quantum wire are greater than 10^3 V/cm; see S. E. Laux, D. J. Frank, and F. Stern, Surf. Sci. **196**, 101 (1988); K. S. Bergus, G. Roos, and H. van Houten, Phys. Rev. B **37**, 10118 (1988).