# Chaotic Mixing as a Renormalization-Group Fixed Point

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A renormalization-group fixed point is found, corresponding to chaotic mixing in the Rayleigh-Taylor instability problem. The outer envelope of the mixing region, adjacent to the heavy fluid, is dominated by a merger of unstable modes (bubbles of light fluid) and dynamically changing length scales. A statistical model is introduced as an approximation to the full two-fluid Euler equation to describe the mixing envelope. Molecular-chaos and continuous-time approximations to this model define an approximate renormalization-group equation, which is shown to have a nontrivial fixed point.

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We construct a renormalization-group fixed point to describe a chaotic mixing process. The Rayleigh-Taylor mixing problem is the study of the dynamic evolution of the interface between two fluids of different densities, subject to a gravitational or accelerating force. In the unstable case, nonlinear modes in the interface grow and a mixing layer develops. Here we study the outer envelope of the mixing layer adjacent to the heavy fluid. For simplicity, we consider a two-dimensional flow, hence a one-dimensional interface. The existence of a renormalization-group fixed point is suggested by the approximate universality of experimentally observed[1,2] and numerically computed[3,4] mixing rates. This Letter presents the first theoretical evidence for this fixed point.

The statistical model results from a simplification of the dynamics defined by the two-fluid Euler equation. The state space of the model is the set of piecewise constant functions on the line, each constant piece representing a single bubble. We are interested in a "molecular-chaos" version of the model, in which nearest-neighbor correlations between adjacent bubbles are ignored. (Possible limitations to this hypothesis have been discussed.[5]) At this level of description, the state space is an ensemble of bubbles whose heights are defined by a probability measure on the line. As we will see below, this measure must be concentrated on a bounded interval defined by a bubble merger criterion given below. As a further assumption, we take the measure to be a uniform distribution of an interval $[-a(t)+\bar{h}(t), a(t)+\bar{h}(t)]$. Uniform distribution on an interval is not preserved dynamically, and so we modify (approximate) the dynamics to preserve this assumption. For simplicity, we assume that the bubbles have a common radius $r(t)$ at time $t$. We are interested in the variance $\sigma(t)$ of this ensemble, defined by the equation $\sigma^2=\int(h-\bar{h})^2 dh/2a$. Thus $a=\sqrt{3}\sigma$. The essence of the molecular-chaos approximation is to consider pair interactions by drawing two adjacent bubbles randomly from the ensemble, or equivalently to regard nearest-neighbor pairs as having uncorrelated heights $h_1$ and $h_2$.

The dynamics of the statistical model, as explained below, treats bubble merger as an event which is discrete in time. We approximate this discrete dynamics by continuous dynamics, and obtain a differential equation for $r$, $\bar{h}$, and $\sigma$. These equations are most conveniently studied in terms of scaled variables. Following conventional scaling assumptions, we let $h'=(h-\bar{h})/r$ and $\sigma'=\sigma/r$. We introduce $h'_m$ as the scaled height separation of adjacent bubbles at which merger occurs and $t_m=t_m(h'_2-h'_1)$ as the time to merger for adjacent bubbles with initial height separation $h'_2-h'_1$, where $h'_2\geq h'_1$. Let $g$ be the gravitational acceleration and $A$ be the Atwood number. Then $dt'=(Ag/r)^{1/2}dt$ is the scaled differential time and $v'=(Agr)^{-1/2}v$ is the scaled velocity. Also $t'_m=(Ag/r)^{1/2}t_m$ is the scaled time to merger. Thus $t'_m(h'_m)=0$. It is convenient to set $t'_m(h')=\infty$ for $h'<0$, and $t'_m(h')=0$ for $h'>t'_m$. $h'_m$ and $t'_m$ determine the dynamics in the approximation we are considering, as we explain in detail.

The dynamics of the statistical model is that at merger height separation $h'_m$ the higher bubble doubles in size (merges with its neighbor) while the lower bubble is removed from the ensemble. Before merger, each bubble moves with a scaled velocity $v'=v'_b+v'_e$, which is a sum of a scaled single bubble velocity $v'_b$ and an envelope velocity $v'_e$.[3,4] This expression for the bubble velocity represents an essential modification of the original Sharp-Wheeler model,[6] because it eliminates a phenomenological parameter from that model, and leads to a renormalization-group fixed point. In the spirit of the molecular-chaos approximation, we ignore here the fact that each bubble has two neighbors. Here $v'_b$ is an absolute number, the single bubble terminal velocity. It is nearly constant and depends only weakly on two dimensionless parameters, the Atwood number $A$ and the compressibility $M^2$ of the two-fluid Euler equation, while $v_e' = v_e' \times (h_2' - h_1')$ depends on $h_2' - h_1'$ in addition.[5,7]

Let $a' = a/r$, $\langle F(h') \rangle = \int F(h') dh'/2a'$, and $\langle F(h_1', h_2') \rangle = \int F dh_1' dh_2'/(2a')^2$. In this notation, the equation

$$dr/dt = (Ag)^{1/2} \langle 1/t_m' \rangle r^{1/2} \tag{1}$$

expresses the hypothesis, in the continuous-time approximation, that the higher bubble from a nearest-neighbor pair doubles in size at the end of its merger time interval. This scaling is based on a picture of the merger process in which the radii of the two neighboring bubbles are frozen until merger is completed. Thus the time-dependent radius $r(t)$ applies to the ensemble, but not to the pair of bubbles selected from the ensemble for merger. The same picture is used below to determine the bubble velocities.

The foregoing discussion, which specifies the statistical model and the approximations, leads to the fundamental dynamical equation for $dh'/dt'$. This equation includes a contribution from $v_b'$ and a statistical contribution from $v_e'$. The dynamics for $h'$ can be formulated either through its action on the stochastic variable $h'$ or, by duality, on the probability measure which defines the distribution of $h'$. It is convenient to use both of these formulations, for distinct terms in the equations. Let $G$ be a function of $h'$ and let $F = \partial G/\partial h'$. The result is

$$\left\langle \frac{dG}{dt'} \right\rangle = v_b' \langle F \rangle + \left\langle \frac{h_m' - h_2' + h_1'}{2t_m'(h_2' - h_1')} F(h_2') \right\rangle - \left\langle G(h_1') \int \frac{dh_2'}{2a' t_m' (h_2' - h_1')} \right\rangle + \left\langle \frac{1}{t_m'} \right\rangle \langle G(h_1') \rangle .$$

The first term is the uniform drift in $h'$ from the single bubble velocity $v_b'$; it is straightforward. The remaining terms derive from the envelope velocity $v_e'$. The second term is the contribution from merger for a bubble that is higher than its neighbor. It is expressed in terms of the action of the dynamics on the stochastic variable $h'$. Merger, *per se*, has no effect on the height of the higher (surviving) bubble. However, the instantaneous envelope velocity $v_e'$, applied to the higher bubble, time averaged from the present to the time of merger, can be computed as $\frac{1}{2}\Delta$(height separation)/$\Delta$(time). This time-averaged instantaneous scaled envelope velocity is the fraction $(h_m' - h_2' + h_1')/2t_m'(h_2' - h_1')$ in the second term.

The third and fourth terms are the contributions to merger for a bubble below its neighbor. In these terms the dynamics is expressed in terms of its action on the measure. The third term corresponds to removal of the bubble from the ensemble and the fourth to its replacement with a bubble drawn at random from the ensemble. Consider the third term. The integral over $h_2'$ is the probability per unit time for a bubble of height $h_1'$ to be removed from the ensemble by merger. This is then integrated over all bubbles in the ensemble, i.e., over $h_1'$. The fourth term adds back a bubble in a probabilistic sense. Since $\langle 1/t_m' \rangle$ is the probability per unit (scaled) time for removing a bubble of height $h_1'$, integrated over $h_1'$, i.e., the total probability per unit time of removing a bubble, the fourth term establishes conservation of probability. For all terms, the envelope dynamics is not given by the instantaneous velocity $v_e'$ combined with merger dynamics, but instead this velocity is time averaged up to and including merger. This is the continuous-time approximation, used also in (1).

We take $G = h/r$ and $F = I$ and obtain

$$\frac{d(\bar{h}/r)}{dt'} = v_b' + \frac{1}{2} \left\langle \frac{h_m' - h_2' + h_1'}{t_m'(h_2' - h_1')} \right\rangle - \left\langle \frac{h_1'}{t_m'(h_2' - h_1')} \right\rangle . \tag{2}$$

Similarly, with $G = (h')^2/2$ and $F = h'$, we have

$$\frac{d\sigma'}{dt'} = \frac{1}{\sigma} \left[ \left\langle h_2' \frac{h_m' - h_2' + h_1'}{t_m'(h_2' - h_1')} \right\rangle - \left\langle \frac{(h_1')^2}{2} \int \frac{dh_2'}{2a' t_m' (h_2' - h_1')} \right\rangle + \frac{(\sigma')^2}{2} \left\langle \frac{1}{t_m'} \right\rangle \right] . \tag{3}$$

The natural interval for $\sigma'$ is given by $0 \le a'(t) \le h_m'/2$. The lower limit is the trivial fixed point corresponding to an unstable interface consisting of bubbles of identical height. At the upper end point, defined by instantaneous merger for bubble pairs with extreme separation, the renormalization-group equation directs the flow into the interval. We show that the right-hand side of (3) is positive at the lower end point, discontinuous at the upper end point, and negatively infinite for $a'(t) > h_m'$, i.e., above the upper end point. The existence of a nontrivial renormalization-group fixed point follows. The undesirable discontinuity at the upper end point is probably related to the artificial choice of a uniform density for the height distribution.

At the upper end point, $t_m' = 0$ and $\langle 1/t_m' \rangle = \infty$. We note that in a one-sided neighborhood of the upper end point, $h_m' - h_2' + h_1' = O(t_m'(h_2' - h_1'))$, since the scaled envelope velocity is bounded. Above the upper end point, i.e., for $a'(t) > h_m'$, this term is zero. Thus the first term in (3) is bounded as $a' \to h_m'/2$. The second and third terms have a negatively divergent integrand in this limit, proportional to $-(a')^2 + (\sigma')^2 = -2(\sigma')^2$; the integral is, however, finite. Above the upper end point, the integral is determined by the set of instantaneous merger, and is negatively infinite. Thus the behavior of (3) at the upper limit is as asserted.

At the lower limit, $t_m'$ is proportional to $-\ln(h_2' - h_1')$

and thus to $-\ln\sigma'$. This asymptotic property follows directly from the linear theory of exponential growth, valid for small disturbances, as applied to the bubble envelope. The second and third terms are thus $O(\sigma' |\ln\sigma'|^{-1})$, and are negligible in comparison to the first term, which is $O(|\ln\sigma'|^{-1})$, due to the term $h'_m = O(1)$ in the numerator. The first term contains contributions of both signs, but is dominantly positive, since $t'_m$ is supported on the set $h'_2 \geq h'_1$. This proves our assertion concerning the lower limit.

Next, we discuss the properties of the fixed-point solution, and its relation to available experimental and computational evidence concerning the bubble envelope. Let $\langle \cdots \rangle_{\sigma^*}$ denote the expectation evaluated at the renormalization-group fixed point $\sigma^*$. Solving Eqs. (1) and (2) with $\sigma' = \sigma^*$ yields $r(t) = (Ag/4)\langle 1/t'_m \rangle^2_{\sigma^*} t^2$, and

$$\bar{h}(t) = \frac{Ag}{2}\left[v'_b + \left\langle \frac{h'_m - h'_2 - h'_1}{2t'_m(h'_2 - h'_1)} \right\rangle_{\sigma^*}\right]\left\langle \frac{1}{t'_m} \right\rangle_{\sigma^*} t^2 . \quad (4)$$

An extensive body of experiment,[1,2] computation based on the full two-fluid Euler equation[2-4,8] and a computation[5,9,10] based on theoretical models,[6,9] predicts that $\bar{h} = \alpha A g t^2$ (constant acceleration), and comparison with (4) shows that in our model[11]

$$\alpha = \frac{1}{2}\left[v'_b + \left\langle \frac{h'_m - h'_2 - h'_1}{2t'_m(h'_2 - h'_1)} \right\rangle_{\sigma^*}\right]\left\langle \frac{1}{t'_m} \right\rangle_{\sigma^*} . \quad (5)$$

Computations[3,4] of the value of $\alpha$ by the front-tracking method, based on the Euler equation, agree with experiment,[1,2] within experimental accuracy, beyond the time of one bubble merger. According to the above results, the constant $\alpha$ appears to be at least approximately universal.

The scaling nature of the bubble merger process and the resulting interface dynamics, which involves all length scales on a dynamically increasing basis, and the apparent universality of $\alpha$ which describes this dynamics, suggest the occurrence of a renormalization-group fixed point as a consistent explanation of the phenomena. We note[5,7] that $v'_b$ is independent of $A$ and $g$ and depends only weakly on the compressibility $M^2$. It would be possible[7] to explore the other term in (5) and the factor $\langle 1/t'_m \rangle_{\sigma^*}$ to see if they also have a universal character.

Our results show that constant acceleration for $\bar{h}$ is a direct consequence of simple scaling and the existence of a renormalization-group fixed point. We obtain a purely theoretical explanation of constant acceleration and the first direct evidence in support of a renormalization-group fixed point to describe the outer envelope of the mixing region for a Rayleigh-Taylor unstable interface. The implications of universality for a fixed point go beyond the value of $\alpha$. Other aspects of the fixed-point ensemble, such as the variance of the bubble heights, would then be expressible in a universal fashion. Such conjectures, easily derivable from this model, can be tested directly against experiment and numerical simulation. The full statistical model could be solved numerically, without approximations, and would provide a check on the approximations made here as well as giving predictions concerning higher moments.

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