Model of a Fragmentation Process and Its Power-Law Behavior

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A simple model for understanding the qualitative features of a fragmentation process is proposed. At a particular point of a tuning parameter of this model, the fragment-size distribution has a scale-invariant power-law behavior. At relatively low values of this parameter, evaporation modes dominate and at high values of the parameter, multifragmentation takes place. The model is exactly soluble in all regimes of the tuning parameter. A unifying connection between evaporation and multifragmentation is found.

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Many phenomena in nature show a power-law behavior in the distribution function of some quantity. A few examples in physics are the size distribution of meteorites,[1] the distribution of cluster sizes at the percolation threshold,[2] droplet sizes at a critical point,[3] and sandpile slides at a self-organized critical point.[4] Power laws in energy and in frequency are also observed, such as in $1/f$ noise. In other areas of endeavor similar behavior has been noted, such as in the frequency distribution of words in a book when each word is ordered such that $N=1$ is the most frequent word, $N=2$ is the next most frequent word, etc. The frequency distribution then falls as $1/N$.[5]

In many cases the power law is a tabulated or empirically observed property of the system whose origin is not well understood. The purpose of this Letter is to present a model in which an exact power law is obtained at a critical point in the value of some tuning parameter, called x. Moreover, the model is exactly soluble for all values of x. As the parameter is varied from small to large values of it, a single and simple expression shows a system passing from evaporation modes to scale-invariant behavior to total multifragmentation. The model presented here arose out of some studies of the multifragmentation of nuclei[6] in heavy-ion collisions where a power law was also seen experimentally.[7] An application of the model to nuclear fragmentation will be given at the end of this paper, where x will be related to thermodynamic variables such as volume V and temperature T. The model is then an exactly soluble canonical ensemble model of nuclear fragmentation. This model may have applications for understanding similar behavior in other systems.

The model considers the partitioning of A objects into groups with n_j composites or clusters of j elements and $j=1,2,\ldots,A$, where $A=\sum_j n_j$. The partitioning of A objects into such groupings can be related to a well-known problem in number theory[8] which is the decomposition of an integer A into integer summands such as

$$5=4+1=3+2=3+1+1=2+2+1$$
$$=2+1+1+1=1+1+1+1+1.$$

The notation $\pi_A=(1^{n_1},2^{n_2},\ldots,j^{n_j},\ldots)$ is used to specify the decomposition. 2 will represent a composite made of two elements so that $5=3+2$ is the decomposition of 5 into two composites, one with two elements, the other with three elements. Such partitioning also appears in the classification of permutations by cycle classes, with a cycle class being specified by $(n_1,n_2,\ldots,n_j,\ldots)=\{n_j\}$, which has n_1 unit cycles, n_2 cycles of length two, etc. The multiplicity m of a given fragmentation, partition, or decomposition is $m=\sum_j n_j$. The total number of partitions of A is given asymptotically by the Hardy-Ramanujan result[8] $P(A)=\exp(\pi\sqrt{2}A/3)/4A\sqrt{3}$. Each possible partition, specified by $\{n_j\}$, is then given a certain weight and the distribution of clusters is obtained by ensemble averaging n_j over all partitions π_A using this weight. Various possibilities for weights will now be given. Different choices define different models of fragmentation. One weight will be shown to lead to an exponential falloff in the cluster distribution with size k while another leads to a power-law behavior.

The first assumption to be made is that every partition is equally likely. It was noted by Sobottka and Moretto[9] that the frequency of clusters of size k, $Y_A(k)$, is $Y_A(k)=P(A-k)+P(A-2k)+\cdots$. Then, using the Hardy-Ramanujan result for $P(A-k)$, etc.,

$$Y_A(k)=\frac{1}{4\sqrt{3}A}\frac{1}{\exp[(\pi/2)(2/3A)^{1/2}k]-1}$$
$$\sim\frac{\exp[-(\pi/2)(2/3A)^{1/2}k]}{4A\sqrt{3}}. \quad (1)$$

This Bose-Einstein-like distribution is the result obtained by Aichelin and Hufner[10] in their description of fragmentation.

Another situation arises by considering a microstate counting factor for each partition. One weight which gives an exact power-law result is an assignment M_2 for each partition where

$$M_2(\{n_j\};A)=M_2(n_1,n_2,\ldots;A)$$
$$=\frac{A!}{\prod j^{n_j}n_j!}=\frac{A!}{1^{n_1}n_1!2^{n_2}n_2!\cdots A^{n_A}n_A!}. \quad (2)$$

$n_k!$'s are Gibbs factorials. M_2 represents the number of permutations of A objects into a particular cycle class $(n_1,n_2,\ldots,n_j,\ldots)$. The probability of a given partition, cycle class, or fragmentation is then $P(n_1,n_2,\ldots)=M_2(n_1,n_2,\ldots;A)/A!$. For this weight the frequency of k is then an exact $1/k$ power law:

$$Y_A(k)=\sum_{\pi_A} n_k P(n_1 n_2,\ldots;A)=1/k \tag{3}$$

and is independent of A, i.e., of any scale set by A except the trivial constraint $k\le A$ which is not the case of Eq. (1) where $\sqrt{A}$ sets the scale. It should also be noted that this simple result has the same functional form of Zipf's law,[5] both having a hyperbolic form. The result of Eq. (3) also has the same functional behavior of the sandpile distribution function $D(s)$ given in Ref. 4 where s is the size of the slide, i.e., $D(s)\sim s^{-1}$ at the self-organized critical point.

In the next level of complexity, a tuning parameter x is introduced into the weight function. Specifically, let

$$W_A(\{n_j,x_j\})=M_2\prod_{j=1}^{A} x_j^{n_j}, \tag{4}$$

with M_2 given by Eq. (2) and let $W_A(\{x_i\})=\sum_{\pi_A} W_A(\{n_j,x_j\})$, which is also the cycle indicator of the permutation group.[8] The generating function for $W_A(\{x_j\})$ is

$$\exp\left[ux_1+u^2\left(\frac{x_2}{2}\right)+u^3\left(\frac{x_3}{3}\right)+\cdots\right]=\sum_{A=0}^{\infty} W_A(\{x_j\})\frac{u^A}{A!}, \tag{5}$$

which plays the role of the grand-canonical ensemble partition function. A very interesting result follows when all $x_j=x$. With $m=\sum n_j$, $W_A(\{n_j\},x)=M_2(\{n_j\})\times x^m$, and when $M_2(\{n_j\})$ is summed over all partitions with a given m, the resulting quantity is the signless Stirling number of the first kind $(-1)^{A-m}S_A^m$. They are generated from a fractorial-moment generating function

$$x(x+1)(x+2)\cdots(x+A-1)=\sum_{m=1}^{A}(-1)^{A-m}S_A^m x^m$$

and they satisfy a recurrence relationship $S_{n+1}^m=S_n^{m-1}-nS_n^m$. The normalized probability of a given fragmentation is now $M_2(\{n_j\})x^m/x(x+1)\cdots(x+A-1)$. The role of the tuning parameter x is as follows: The M_2 combinatorial factor favors low-multiplicity events with evaporation $(n_1=1, n_{A-1}=1)$ having the highest M_2 weight and the fused system $(n_A=1)$ the next most highest M_2 weight. For $x\ll 1$, the fused mode $(m=1)$ has one power of x, while the evaporation mode $(m=2)$ has x^2, so that the fused mode dominates. As x increases the weight function shifts the mode of decay into evaporation and other $m=2$ situations. With still larger x the system fragments into other complex fragmentation schemes. At $x=1$, the frequency distribution of the fragmentation scheme, as measured by its mass distribution, is given by a scale-invariant power law [Eq. (3)]. At $x\gg 1$, complete dissociation becomes the dominant mode of decay. The parameter x must therefore involve binding-energy, temperature, level-density, and volume effects, and this dependence is given below.

The result for the frequency of k at any value of x can be found analytically. Letting this quantity be $Y_A(k,x)$ and using the generating function $W_A(\{x_j\})$ for $W_A(\{n_j,x_j\})$ and the result of Eq. (5), the following exact result is obtained:

$$Y_A(k,x)=\frac{1}{k}\left(\frac{A!}{(A-k)!}\frac{\sum_{m=1}^{A-k}(-1)^{A-k+m}S_{A-k}^m x^{m+1}}{x(x+1)(x+2)\cdots(x+A-1)}\right)=\frac{1}{k}\left(\frac{A!x\Gamma(x+A-k)}{(A-k)!\Gamma(x+A)}\right), \tag{6}$$

where the Γ functions are given by $\Gamma(z+1)=z!$. For $k=A$, the sum in Eq. (7) is to be taken as x. The result of Eq. (6) is obtained by differentiating these generating functions with respect to the x_j's. At $x=1$, the scale-invariant power law of Eq. (3) follows. At $x=1$, the mean multiplicity $m\sim\ln A+\gamma$, where $\gamma=0.57721$ is Euler's number; $\ln A\sim 5.3$ for $A=200$. Putting $x=1+\epsilon$ generates the critical-point behavior. In particular, when $x=1+\epsilon$ the coefficient of ϵ^n in the numerator of Eq. (6) is $B_n(A-k)+B_{n-1}(A-k)$, where B_n is a binomial moment of the multiplicity distribution. The resulting yield near $x=1$ can be shown to be

$$Y_A(k)=\frac{1}{k}\frac{1+\epsilon g_A(k)+\theta(\epsilon^2)}{1+\epsilon(1+\frac{1}{2}+\frac{1}{3}+\cdots+1/A)}\sim\frac{1}{k}\exp\left(-\frac{\epsilon k}{A}\right). \tag{7}$$

While the first form of $Y_A(k)$ is exact to order ϵ, the exponential form is valid only for $k\ll A$ with $\epsilon=x-x_c$ and $x_c=1$. ϵ is similar to the $p-p_c$ of percolation models.[2] $g_A(k)$ is given by $g_A(k)=1+[1+\frac{1}{2}+\frac{1}{3}+\cdots 1/(A-k)](1-\delta_{k,A})$ so that $g_A(A)=1$, $g_A(A-1)=1+1$, $g_A(A-2)=1+1+\frac{1}{2}$, etc. At $\epsilon=0$, $\sum_k k^2 Y_A(k)=A\times(A+1)/2\sim A^2/2$. For any ϵ or x, $A=\sum_k kY_A(k)$. Also for $\epsilon=0$, $\langle n_1^2\rangle=1$ at $A=1$, $\langle n_1^2\rangle=2$ for all other A, and $\langle n_1 n_r\rangle=1$ for $r\ne A\ne 1\ne r$.

When the tuning parameter is very large, $x\gg A$, the results are

$$Y_A(k,x\gg A)\sim\frac{A!}{(A-k)!}\frac{1}{k}\frac{1}{x^{k-1}}\sim\frac{A}{k}\left(\frac{A}{x}\right)^{k-1},\qquad Y_A(1,x\gg A)\sim A. \tag{8}$$

The result $Y_A(1,x\gg A)\to A$ is the condition for com-

plete fragmentation into individual pieces and the mean multiplicity $m \to A$. At the other extreme of small values of the tuning parameter, $x \ll 1$, and in particular for $x < 1/(\gamma + \ln A)$, $Y_A(k,x)$ is given, to lowest order in x, by

$$Y_A(A, x \ll 1) \sim 1 - (\ln A + \gamma)x ,$$
$$Y_A(k, x \ll 1) \sim xA/k(A-k) . \tag{9}$$

At $x = 0$, $Y_A(A,0) = 1$ and only the fuse mode is present. $Y_A(k, x \ll 1)$ for $k \neq A$ is a U-shaped behavior in k with a minimum at $k = A/2$. Note that evaporation of a unit produces one cluster of size 1 and another of size $A-1$. For $x \ll 1$, the fused mode dominates and the mean multiplicity is ~ 1. For x between $1/(\ln A + \gamma)$ and $2/(\ln A + \gamma)$ the $m \sim 3$ modes dominate and their distribution is asymmetric with k. The contribution of different m modes can be found from

$$S^n_{A-k} = (A-k)![\gamma + \ln(A-k)]^{n-1}/(n-1)! .$$

Note also that the multiplicity shifts from $m \sim 2$ to $m \sim \ln A$ to $m \sim A$ as the transition is made from evaporation to power behavior to total fragmentation.

At $x = 1$, the multiplicity distribution is given by the signless Stirling numbers of the first kind. Moments of the probability distribution for the multiplicity distribution $P(m,A) = (-1)^{A-m} S^m_A / A!$ can be easily obtained and, in particular, factorial moments defined by $F_n = \sum_m m(m-1)\cdots(m-n+1)P(m,A)$ are simply evaluated $F_n = n! C^{n+1}_A / A!$, where $C^m_A = (-1)^{A-m} S^m_A$. As an example, F_1, the mean multiplicity, is given by $F_1 = 1 + \frac{1}{2} + \cdots + 1/A \sim \ln A + \gamma$ for large A. The mean size of a cluster is then $\sim A/\ln A$. Higher moments can also be obtained from the recurrence relationship[8] $B_n(A) = B_n(A-1) + A^{-1} B_{n-1}(A-1)$, where the binominal moments are $B_n = (1/n!)F_n$. The variance is $V_2(A) = \frac{1}{4} + \frac{2}{9} + \cdots + (A-1)/A^2$. Other weight functions (not normalized) of the form $W(\{n_j\}) = M_2(\{n_j\}) x^m y^{n_1}$ are also soluble and such results will be considered elsewhere.

The present model can be used to study general features of cluster distributions produced in various processes in different areas such as those discussed at the beginning of this Letter. A specific application will now be given to nuclear fragmentation as an illustration. In the nuclear case, a weight W for each fragmentation $(\{n_j\})$ can be specified by the exponential of the entropy S associated with that fragmentation, $W \sim \exp(S/k_B)$. A model for the entropy is then made and one possibility[11] is the Sakur-Tetrode expression:

$$S = \sum_j k_B n_j \ln\{[V/v_0(j)] e^{5/2} Z_{\rm int}(j)/n_j\} . \tag{10}$$

V is the volume of the system and $v_0(j) = h^3/(2\pi M_j k T)^{3/2}$. $Z_{\rm int}(j) = \exp[E_B(j)/k_B T] \sum \exp(-E^*/k_B T)$ is the internal partition function of a cluster of size j. The sum is over excited states E^*. A weight function of the form of Eq. (4) with $x_j = x$ evolves when the binding energy $E_B(j)$ is simply taken to be a backshifted form $E_B(j) = a_B(j-1)$. For this choice, the $j=1$ monomer has no binding and $E_B(j)/j$ saturates at a_B. Also light clusters have a reduced $E_B(j)/j$. The excited-state partition function in a Fermi-gas model[12] has an exponential part $\sim \exp[a(j)/k_B T]$, where $a(j) \sim j/\epsilon_0$ with ϵ_0 the level-density parameter. Also, $a(j)$ is then backshifted to suppress the $j=1$ monomer case so that $a(j) = (j-1)/\epsilon_0$. The Fermi-gas result can be cut off by setting $T = TT_0/(T+T_0)$ as discussed in Ref. 13. The tuning parameter can be shown to be of the form

$$x = [V/v_0(1)] e^{-a_B/k_B T} f_{\rm int} \tag{11}$$

by comparing the entorpy weight $W \sim e^{S/k_B}$ with that of $M_2 x^m$. $f_{\rm int} \sim \exp[-k_B T T_0/\epsilon_0(T+T_0)]$ for a cutoff Fermi gas. To obtain the result of Eq. (11) use was made of the constraint $\sum_j n_j = A$ and the definition of $m = \sum n_j$.

Now substituting this choice of x into Eq. (9) gives a result in Fermi's monograph *Thermodynamics*[14] for the evaporation of particles $(k=1)$ into a volume V which reads $N = g_s[V/v_0(1)]\exp(-W_f/k_B T)$, where g_s is the spin-degeneracy factor and N is the number of particles evaporated. W_f is the work function. a_B equals the separation energy or work function W_f when $E_B(j) = a_B \times (j-1)$. The factor $\exp(-a_B/k_B T)$ acts as a barrier against evaporation. Using this same x, the result of Eq. (8) is just a form of the Saha equation.[15] The Saha equation and Fermi's evaporation equation are limiting forms of one underlying equation, Eq. (6). The Saha equation was used in Ref. 11 to discuss composite-particle formation in high-energy nuclear collisions. The volume V is the freeze-out volume[11] and this volume represents the largest volume over which equilibrium is maintained. This volume can be obtained from properties of the distribution of composite particles or by resorting to a Hanbury-Brown–Twiss correlation measurement. The equilibrium volume in high-energy nucleus-nucleus collisions is about $4V_N$, where V_N is the normal volume of A nucleons. The barrier suppression factor for evaporation becomes the Boltzmann binding-energy enhancement factor of the Saha equation in these limiting forms of $Y_A(k,x)$. Comparing Eqs. (8) and (9), the reader will note that the role of x moves from a denominator for multifragmentation $(1/x^{k-1})$ to the numerator in evaporation. Thus

$$\exp(-a_B/k_B T) \to 1/\exp(-a_B/k_B T)^{k-1}$$
$$= \exp[E_B(k)/k_B T] .$$

Finally, given the weight for each partition and the expression for x other quantities associated with nuclear fragmentation can be evaluated. Here, the focus was on the cluster-size distribution function. The multiplicity, fluctuations, and correlations are also easily evaluated in

terms of the given x of Eq. (11) and such results will be presented in a future publication. Other approaches to nuclear fragmentation can be found in Refs. 10, 13, and 16–19.

The purpose of this Letter was to develop a simple and exactly soluble model for a fragmentation process which hopefully captures the essence of various features of fragmentation. The model gives an exact power-law behavior in the cluster-size distribution function at a particular point of a tuning parameter. Moreover, a single expression seems to contain a whole spectrum of fragmentation schemes from the simplest to the most complex. At low values of the tuning parameter, evaporation modes dominate and for large values of the parameter, multifragmentation takes place. The tuning parameter was then related to thermodynamic variables for a particular application of the model to nuclear fragmentation. Work-function barriers against evaporation are shown to evolve into binding-energy enhancement factors of the Saha equation and a unifying connection is found between these limits.

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