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Geometric Phase in the Classical Continuous Antiferromagnetic Heisenberg Spin Chain

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We show that the time evolution of a space curve is associated with a geometric phase. This phase arises from the path dependence of the rotation of the natural Frenet-Serret triad with respect to a nonrotating (Fermi-Walker) frame. We derive a general expression in 1+1 dimensions for the phase and the associated gauge potential, and discuss the application of this formalism to the classical, continuous, antiferromagnetic Heisenberg spin chain.

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The study of space curves[1] finds applications in many areas of physics. As illustrative examples of a space curve we have (i) a thin nonstretching vortex filament in a fluid,[2] (ii) a spin configuration in a classical ferromagnetic chain[3] (the constant-magnitude spin vectors being the tangents along a curve), and (iii) a twisted optical fiber.[4] Certain dynamical properties of a space curve have been studied by Lamb,[5] where it is shown that the evolution of some simple types of curves can be related to nonlinear partial differential equations associated with soliton[6] propagation.

In this Letter we investigate another aspect of the space-curve formalism by deriving an expression for the underlying *geometric phase* associated with the time evolution of the curve. The study of such geometric phases (introduced in recent literature by Berry[7]) and the corresponding gauge fields has gained attention in a wide spectrum of problems in classical and quantum physics.[8] Our purpose is to present a general formalism in 1+1 dimensions in relation to moving space curves and apply it to the *continuum* version of the (classical) antiferromagnetic chain. This is especially useful in clarifying certain points in recent literature[9,10] on the geometric phase in the antiferromagnetic chain. Furthermore, we point out that if explicit solutions of the space-curve evolution equation can be found (which is possible in many applications such as spin chains), our derivation also makes it possible to investigate under what conditions a class of solutions like the solitons would imply the presence of a nonvanishing phase or gauge field. Such insights will be useful for a better understanding of the nature of possible topological features in interacting spin systems and their excitation spectra. Finally, our formalism could also find other applications such as the evolution of polymer chains and strings.

At a given instant of time, say, u_0, a spaces curve[1] is described by its natural equations: $\kappa=\kappa(s)$, $\tau=\tau(s)$, where κ, τ, and s are the curvature, the torsion, and the length (treated as a natural parameter) of the space curve. We denote by $\mathbf{t}$ the unit tangent vector to the curve and by $\mathbf{n}$ and $\mathbf{b}$ its principal normal and binormal, respectively. They are related by the Frenet-Serret formulas:

$$\mathbf{t}_s=\kappa\mathbf{n},\quad \mathbf{n}_s=-\kappa\mathbf{t}+\tau\mathbf{b},\quad \mathbf{b}_s=-\tau\mathbf{n}\,, \tag{1a}$$

where the subscript s denotes d/ds. We also have

$$\kappa^2=\mathbf{t}_s\cdot\mathbf{t}_s \quad\text{and}\quad \tau=\mathbf{t}\cdot(\mathbf{t}_s\times\mathbf{t}_{ss})/\kappa^2\,. \tag{1b}$$

As time evolves κ and τ, and thus $\mathbf{t},\mathbf{n},\mathbf{b}$, are in general functions of both s and u, i.e., $\kappa=\kappa(s,u)$, $\tau=\tau(s,u)$.

For a fixed s the functions

$$\kappa_0^2(u)=\mathbf{t}_u\cdot\mathbf{t}_u \quad\text{and}\quad \tau_0(u)=\mathbf{t}\cdot(\mathbf{t}_u\times\mathbf{t}_{uu})/\kappa_0^2\,,$$

where the subscript u denotes d/du, represent, respectively, the curvature and the torsion of a new space curve with u as its natural parameter. Introducing the Darboux vector $\boldsymbol{\xi}=\tau\mathbf{t}+\kappa\mathbf{b}$, Eqs. (1) can be written as

$$\mathbf{t}_s=\boldsymbol{\xi}\times\mathbf{t},\quad \mathbf{n}_s=\boldsymbol{\xi}\times\mathbf{n},\quad\text{and}\quad \mathbf{b}_s=\boldsymbol{\xi}\times\mathbf{b}\,. \tag{2}$$

The Darboux vector $\boldsymbol{\xi}$ plays the role of an angular velocity of the Frenet-Serret triad. So, e.g., **n** and **b** rotate around **t** with angular velocity τ. Let us consider a nonrotating frame in the moving plane spanned by **n** and **b** using the usual Fermi-Walker parallel transport along the curve[11]

$$\frac{DA^i}{ds}=\{\kappa \mathbf{b}\times\mathbf{A}\}^i . \tag{3}$$

As we move from s_0 to s_1, a phase $\Phi_1=\int_{s_0}^{s_1}\tau(s)ds$ develops between the natural frame (**n** and **b**) and the nonrotating frame. As we move from u_0 to u_1 along the "temporal" space curve, a phase $\Phi_0=\int_{u_0}^{u_1}\tau_0(u)du$ develops between the natural frame and the corresponding nonrotating frame.

We now consider the space-time evolution of the tangent to the moving space curve from the point $a=(s,u)$ to the point $d=(s+\Delta s,u+\Delta u)$ using paths (A) and (B), shown in Fig. 1. As is clear from Fig. 1 the rotation angle Φ is given in the two cases by

$$\begin{aligned}\Phi_1&=\tau(s,u)\Delta s+\tau_0(s+\Delta s,u)\Delta u ,\\ \Phi_2&=\tau_0(s,u)\Delta u+\tau(s,u+\Delta u)\Delta s .\end{aligned}\tag{4}$$

The phase difference $\delta\Phi=\Phi_1-\Phi_2$ is

$$\begin{aligned}\delta\Phi&=\left(\frac{\partial\tau_0}{\partial s}-\frac{\partial\tau}{\partial u}\right)\Delta s\Delta u+O(\Delta^3)\\ &=F(s,u)\Delta s\Delta u+O(\Delta^3) ,\end{aligned}\tag{4a}$$

where $F(s,u)=\partial\tau_0/\partial s-\partial\tau/\partial u$ can be thought of as a measure of "anholonomy density" of the system. Thus the total "anholonomy" as the system evolves in time from $u=T_1$ to $u=T_2$ and in space from, e.g., $s=-\infty$ to $s=+\infty$ is

$$\begin{aligned}\Phi&=\int_{-\infty}^{\infty}ds\int_{T_1}^{T_2}du\,F(s,u)\\ &=\left(\int_{T_1}^{T_2}\tau_0(s,u)du\right)_{s=-\infty}^{s=+\infty}\\ &\quad-\left(\int_{-\infty}^{+\infty}\tau(s,u)ds\right)_{u=T_1}^{u=T_2} .\end{aligned}\tag{5}$$

In what follows we find Φ as a function of κ and τ for a moving space curve. To find τ_0 we require $\mathbf{t}_u$ and $\mathbf{t}_{uu}$. These are determined using a procedure suggested by Lamb.[5]

The Frenet-Serret Eqs. (1) combine to give

$$(\mathbf{n}+i\mathbf{b})_s+i\tau(\mathbf{n}+i\mathbf{b})=-\kappa\mathbf{t} .$$

We assume that $\tau(s,u)\to c_0$ (const), for $|s|\to\infty$. Introducing the quantities[5]

$$\mathbf{N}=(\mathbf{n}+i\mathbf{b})\exp\left(i\int_{-\infty}^{s}ds'(\tau-c_0)\right)$$

and

$$q=\kappa\exp\left(i\int_{-\infty}^{s}ds'(\tau-c_0)\right) ,$$

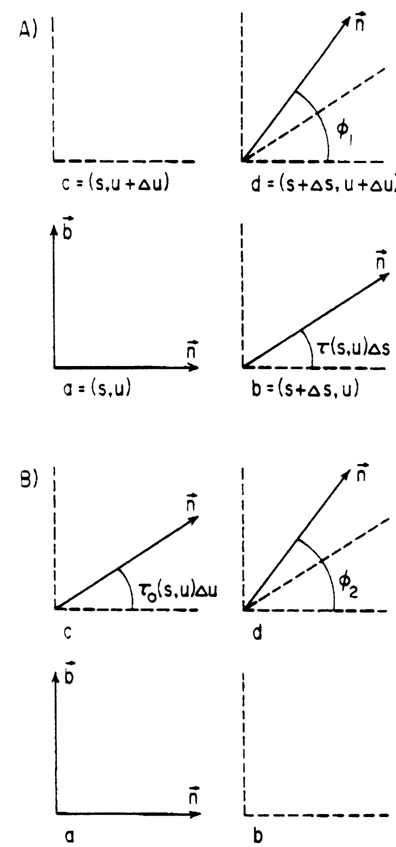

FIG. 1. (a) The route $a\to b\to d$: The phase is $\phi_1=\tau(s,u)\Delta s+\tau_0(s+\Delta s,u+\Delta u)\Delta u$. (b) The route $a\to c\to d$: The phase is $\phi_2=\tau_0(s,u)\Delta u+\tau(s,u+\Delta u)\Delta s$.

we find that $\mathbf{N}_s=-ic_0\mathbf{N}-q\mathbf{t}$ and also $\mathbf{t}_s=\frac{1}{2}(q^*\mathbf{N}+q\mathbf{N}^*)$, which determines the spatial (s) dependence of the natural triad. The temporal (u) dependence is given by

$$\begin{aligned}\mathbf{N}_u&=\alpha\mathbf{N}+\beta\mathbf{N}^*+\gamma\mathbf{t}=iR\mathbf{N}+\gamma\mathbf{t} ,\\ \mathbf{t}_u&=\lambda\mathbf{N}+\mu\mathbf{N}^*+\nu\mathbf{t}=-\tfrac{1}{2}(\gamma^*\mathbf{N}+\gamma\mathbf{N}^*) ,\end{aligned}$$

where we have used the constraints $\mathbf{N}\cdot\mathbf{N}^*=2$, $\mathbf{N}\cdot\mathbf{t}=\mathbf{N}^*\cdot\mathbf{t}=\mathbf{N}\cdot\mathbf{N}=0$, together with the compatibility condition $\mathbf{t}_{su}=\mathbf{t}_{us}$. Using $\mathbf{N}_{su}=\mathbf{N}_{us}$ yields $q_u-\gamma_s+i(c_0\gamma-Rq)=0$ with $R_s=\frac{1}{2}i(\gamma q^*-\gamma^*q)$. The above equation for q can support soliton solutions for particular choices of γ (expressed in terms of q and its derivatives) determined by the time evolution of the spaced curve. On the other hand, using the fact that $\mathbf{t}_u\cdot\mathbf{t}=0$ we can write $\mathbf{t}_u=g\mathbf{n}+h\mathbf{b}$, where g and h are functions of s and u. This allows us to express Frenet-Serret-like equations for the temporal curve,

$$\begin{aligned}\mathbf{t}_u&=g\mathbf{n}+h\mathbf{b} ,\\ \mathbf{b}_u&=\mathbf{n}\left(R-\int_{-\infty}^{s}ds'\tau_u\right)-h\mathbf{t} ,\\ \mathbf{n}_u&=\mathbf{b}\left(\int_{-\infty}^{s}ds'\tau_u-R\right)-g\mathbf{t} .\end{aligned}\tag{6}$$

The curvature and torsion of this curve are given by

$$\kappa_0^2 = \mathbf{t}_u \cdot \mathbf{t}_u = g^2 + h^2 ,$$

$$\tau_0 = \frac{\mathbf{t}\cdot(\mathbf{t}_u \times \mathbf{t}_{uu})}{\mathbf{t}_u \cdot \mathbf{t}_u} = \int_{-\infty}^{s} ds' \, \tau_u - R + \frac{\partial f}{\partial u} ,$$

where $f = \tan^{-1}(h/g)$.

Now, we are ready to calculate the total phase Φ from Eq. (5). We obtain the following expression:

$$\Phi = -\int_{T_1}^{T_2} du \int_{-\infty}^{+\infty} ds \, \kappa h + \left[\tan^{-1}\left(\frac{h}{g}\right)\bigg|_{u=T_2} - \tan^{-1}\left(\frac{h}{g}\right)\bigg|_{u=T_1} \right]_{s=-\infty}^{s=+\infty} . \quad (7)$$

Using $\mathbf{t}_s = \kappa \mathbf{n}$ and Eq. (6), we find that $\mathbf{t}\cdot(\mathbf{t}_s \times \mathbf{t}_u) = \kappa h$ so that

$$\Phi = -\int_{T_1}^{T_2} \int_{-\infty}^{+\infty} \mathbf{t}\cdot(\mathbf{t}_s \times \mathbf{t}_u) ds \, du + f_0 , \quad (8)$$

with f_0 an integration constant. For appropriate boundary conditions, such that $\mathbf{t} \to \mathbf{t}_0$ (a constant vector) at space-time infinity, the first part of (8) becomes the *Pontryagin index* and $f_0 = 0$. Note that $R_s = \kappa h$, $\Phi = -\int du \, R$, and $R = -\frac{1}{2} \mathbf{N}^* \mathbf{N}_u$. This shows that by replacing the complex unit vector $\mathbf{N}/\sqrt{2}$ by a normalized quantum state $|\mathbf{N}(u)\rangle$ and specializing to cyclic evolution, we make a correspondence with Aharanov and Anandan's expression[12] $\beta = i\int \langle N | \partial/\partial u | N \rangle du$ for Berry's phase. However, our expression is applicable to general evolutions.[13]

Let us now determine the corresponding *gauge potential* related to the phase Φ. Consider the following construction. We transport, by Euclidean parallel transport, all the tangent vectors to our "spatial" and "temporal" curves to the center of a unit sphere. The tips of the tangent vectors trace out the spherical images of these space curves on the unit sphere. Now, consider a small plaquette *abcd* on the surface of the unit sphere, where the point *a* corresponds to the point (s,u) of Fig. 1, *b* to $(s+\Delta s, u)$, *c* to $(s, u+\Delta u)$, and *d* to $(s+\Delta s, u+\Delta u)$. We note that the vector $d\mathbf{t}$ is tangent to the spherical images of the space curves. We consider now the following expressions for the phase difference $\delta\Phi$ in terms of the gauge potential:

$$\begin{aligned} \delta\Phi &= \oint \mathbf{A}\cdot d\mathbf{t} \\ &= A_i(s,u)\frac{\partial t_i}{\partial s}\Delta s + A_i(s+\Delta s,u)\frac{\partial t_i}{\partial u}\Delta u \\ &\quad - A_i(s+\Delta s, u+\Delta u)\frac{\partial t_i}{\partial s}\Delta s - A_i(s,u+\Delta u)\frac{\partial t_i}{\partial u}\Delta u , \end{aligned}$$

where $\mathbf{A} = \mathbf{A}(\mathbf{t})$ is the vector potential and $\oint$ is the closed integral over the plaquette *abcd*. Keeping only terms up to second order in Δs and Δu,

$$\begin{aligned} \delta\Phi = \oint \mathbf{A}\cdot d\mathbf{t} &= \left[\frac{d}{ds}\left(\mathbf{A}\cdot\frac{\partial \mathbf{t}}{\partial u}\right) - \frac{d}{du}\left(\mathbf{A}\cdot\frac{\partial \mathbf{t}}{\partial s}\right)\right]\Delta s \Delta u \\ &= (\mathrm{curl}_{\mathbf{t}}\mathbf{A})\cdot(\mathbf{t}_s \times \mathbf{t}_u)\Delta s \Delta u . \end{aligned} \quad (9)$$

Thus, if $\nabla_{\mathbf{t}} \times \mathbf{A} = \mathbf{t}$, we recover our previous expression (8) for the phase Φ. This identifies $\mathbf{A} = \mathbf{A}(\mathbf{t})$ as the vector potential of a unit monopole at the center of the unit sphere. A comparison of Eqs. (9) and (4a) leads to $\mathbf{A}\cdot\mathbf{t}_u = \tau_0$ and $\mathbf{A}\cdot\mathbf{t}_s = \tau$. Using Eqs. (1a) and (6) in these relations gives

$$\mathbf{A} = \left(\frac{\tau}{k}\right)\mathbf{n} + \left[\frac{1}{h}\left(\tau_0 - \frac{g\tau}{k}\right)\right]\mathbf{b} . \quad (10)$$

Let us now consider a particular time evolution of our space curve: such that after a time interval Δu the curve almost returns to its original configuration. In that case the points *a* and *c* and *b* and *d* (Fig. 1) will be very close. Also the paths $a \to b$ and $d \to c$ almost coincide, so that the main contribution to the phase arises from the almost closed loops $a \to c \to a$ and $b \to d \to b$:

$$\delta\Phi = \tau_0(s+\Delta s, u)\Delta u - \tau_0(s,u)\Delta u = \frac{\partial \tau_0}{\partial s}\Delta s \Delta u . \quad (11)$$

In terms of the vector potential this phase is

$$\delta\Phi = \frac{d}{ds}\left(\mathbf{A}\cdot\frac{\partial \mathbf{t}}{\partial u}\right)\Delta s \Delta u . \quad (12)$$

Our discussion [Eqs. (4a)-(8)] leading to the total phase Φ shows that the above "partial" phase alone cannot be associated with the topological term appearing in Eq. (8).

We now consider a particular application of the above general formalism: the one-dimensional antiferromagnetic Heisenberg spin chain.

The continuum approximation in the case of an antiferromagnet is valid only within each sublattice. Introducing the variables $\boldsymbol{\eta}_i = (1/2S)(\mathbf{S}_i - \mathbf{S}_{i+1})$ and $\boldsymbol{\chi}_i = (1/2S)(\mathbf{S}_i + \mathbf{S}_{i+1})$, where $\mathbf{S}_i$ is the spin vector at the site *i*, and specializing to the physically interesting case with $|\boldsymbol{\chi}| \ll |\boldsymbol{\eta}|$, we find the following equation of motion:

$$\boldsymbol{\eta}_u = \boldsymbol{\eta}_x \times \boldsymbol{\eta} . \quad (13)$$

Details of this derivation will be published elsewhere. Since $\boldsymbol{\eta}\cdot\boldsymbol{\eta}_x = 0$ we also have

$$\boldsymbol{\eta}_x = \boldsymbol{\eta} \times \boldsymbol{\eta}_u . \quad (14)$$

Equations (13) and (14) may be written compactly as

$$\partial_\mu \eta^\alpha = \epsilon_{\mu\nu}\epsilon_{\alpha\beta\gamma}\eta^\beta \partial_\nu \eta^\gamma . \quad (15)$$

These equations support several classes of solution, including metastable *instantons* in each homotopic class,[14] and can be derived by minimizing the following Hamiltonian:

$$H = \int\int \left[\left(\frac{\partial \eta}{\partial t}\right)^2 + \left(\frac{\partial \eta}{\partial x}\right)^2\right] dx \, dt . \quad (16)$$

We identify this as the Hamiltonian for the antiferromagnetic chain. Associating $\boldsymbol{\eta}$ with the tangent to a space curve,[3] we may write

$$\boldsymbol{\eta}_u = \boldsymbol{\eta}_s \times \boldsymbol{\eta} = \kappa \mathbf{n} \times \mathbf{t} = -\kappa \mathbf{b} . \tag{17}$$

Comparing expression (17) with the general one above, $\mathbf{t}_u = g\mathbf{n} + h\mathbf{b}$, we identify $g=0$ and $h=-\kappa$. In this case the geometrical phase (8) becomes

$$\Phi = -\int\int \mathbf{t}\cdot(\mathbf{t}_s \times \mathbf{t}_u)\, ds\, du = \int\int \kappa^2\, ds\, du . \tag{18}$$

This is the expression for the phase of a continuous antiferromagnetic Heisenberg spin chain suggested in Ref. 9. An attempt to justify it was reported in Ref. 10. These approaches are based on a particular time evolution of the spin configuration: such that after a time interval Δu the system returns to its original configuration. For that case only the total phase Φ can be constructed from "local Berry phases" for the individual spins, Eq. (11). Namely,

$$\Phi = \int\int \frac{d}{ds} \tau_0\, ds\, du = \int ds \left[\frac{d}{ds} \int \tau_0\, du \right] ,$$

where $\int \tau_0 du$ is the Berry phase for the individual spin at the point s.[7,8,11]

In conclusion, we have shown, for the time evolution of a space curve, how the local anholonomy density related to the rotation of the natural Frenet-Serret frame around the tangent leads to a global topological invariant of the system (the Pontryagin index). The relation of this quantity to the Berry phase has been elucidated: The total geometric phase of an isolated composite system (i.e., with no external parameters) is not simply the sum of Berry phases of the composite parts of the system. Rather it is the sum over relative phases of neighboring components in a moving frame. In some limits this becomes a sum over local Berry phases. The more general situation is an important feature which must be properly treated in discussion of phases in interacting many-body systems like the Heisenberg chain. Thus, for example, the geometric phase of an instanton, which is a solution that returns to its original configuration, appears to be its Pontryagin index (i.e., is quantized). Applying the general formalism for the time evolution of a space curve and using the equations of motion for a continuous Heisenberg antiferromagnet with appropriate boundary conditions, we have shown that its geometric phase is the Pontryagin index, which is in agreement with the fact that this system has instanton solutions in 1+1 dimensions. Possible extensions of our formalism to 2+1 dimensions are under investigation.

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