PHYSICAL REVIEW LETTERS

VOLUME 64 | 23 APRIL 1990 | NUMBER 17

Fractional Statistics on a Torus

Torbjörn Einarsson[(a)]

Institute of Theoretical Physics, Chalmers University of Technology, S-412 96 Göteborg, Sweden

(Received 16 January 1990)

It is shown that fractional statistics on a torus is consistent only with multicomponent wave functions. This is expected to be generally true for all 2D closed, multiply connected, orientable manifolds. The fractional quantum Hall effect with periodic boundary conditions is seen to fit into this new picture, where the quasiholes may be interpreted as a kind of generalized anyons.

PACS numbers: 05.30.−d, 03.65.Fd, 71.45.−d

There is a possibility of exotic so-called *fractional* (or θ) *statistics* in a quantum system of N identical particles moving on a 2D manifold $\mathcal{M}$.[1,2] For particles obeying fractional statistics (anyons), the complex phase change of the wave function upon interchange of two particles $\exp(i\theta)$ is given by neither $\theta=0$, as in the Bose case, nor $\theta=\pi$, as in the case of Fermi statistics. Considering a plane $\mathcal{M}=\mathcal{R}^2$, there are no restrictions on the statistical angle θ. On a sphere, $\mathcal{M}=S^2$, θ must satisfy[3] $\theta=n\pi/(N-1)$, where $0\le n\le 2N-3$. In the thermodynamic limit $N\rightarrow\infty$ there is no restriction on θ. Surprisingly, the situation on a torus, $\mathcal{M}=\mathcal{T}$, seems to be different. Govindarajan and Shankar have found[4] that for solitons in the O(3) nonlinear σ model on a torus, the only allowed scalar statistics are Bose and Fermi. Imbo, Imbo, and Sudarshan[5] have also suggested that for all closed, orientable, two-manifolds $\mathcal{M}\neq S^2$, these are the only possibilities. It then seems that fractional statistics are not allowed if one imposes periodic boundary conditions (PBC) on a system of particles moving on a plane. This appears surprising since we would like to see the thermodynamic limit insensitive to the global topology.

We will show here that the resolution of this apparent paradox is that on a multiply connected manifold $\mathcal{M}$, scalar statistics does not imply scalar quantum theories. Hence, fractional statistics may be obtained by using a multicomponent wave function. It will be shown that this type of fractional statistics is applicable to the quasiholes of the fractional quantum Hall effect (FQHE) with periodic boundary conditions.

The rest of this Letter consists of two parts. The first part reviews basic facts about quantization and braid groups while the second part discusses new results about fractional statistics on a torus.

Basic facts.—The standard procedure for constructing a quantum theory from a classical configuration space Q is to choose the fixed-time quantum state vectors $\Psi(q)$ as functions from Q into the complex numbers $\mathcal{C}$. More generally, one can choose $\Psi(q)$ to be multivalued and to have M components $\{\Psi_m(q)\}$. The only restriction is[6] that when q is taken along closed loops in Q, $\{\Psi_m(q)\}$ must transform according to a M-dimensional unitary representation of $\pi_1(Q)$, the fundamental group of Q, and map back on the same multivalued state vector. The quantization of a classical system is therefore, in general, not unique and for every distinct irreducible unitary representation (IUR) of $\pi_1(Q)$ there is a distinct quantum theory.

For a system of N indistinguishable hard-core particles moving on a manifold $\mathcal{M}$, $Q=Q_N\equiv(\mathcal{M}^N-\Delta)S_N$, where Δ is the subcomplex of all points in $\mathcal{M}^N$ where two or more particle coordinates coincide, and S_N is the permutation group of N symbols.[1,7] Single-valued state vectors in $(\mathcal{M}^N-\Delta)/S_N$ give only Bose statistics, whereas multivalued state vectors give Fermi and fractional statistics. The latter correspond to single-valued state vectors in $\mathcal{M}^N-\Delta$, but with an external statistical gauge field.[1,2,5] The fundamental group of Q_N is known as the N-string *braid group* $B_N(\mathcal{M})$ of the manifold $\mathcal{M}$. The quantum theories with one-component state vectors $\Psi(q)$ are called *scalar quantum theories.* These correspond to 1D IUR's of $B_N(\mathcal{M})$ and give *scalar statistics* where the statistical phases $\exp(i\theta)$ are given by the characters of the 1D representations.[7]

The N-string braid group of the torus $B_N(\mathcal{T})$ is an infinite non-Abelian group defined as the fundamental group of $(\mathcal{T}^N-\Delta)/S_N$. An element of $B_N(\mathcal{T})$ may be thought of as a homotopy class of paths in $\mathcal{T}^N-\Delta$ whose (fixed) initial and final points are related by a permutation of the particle coordinates.

One set of generators of $B_N(\mathcal{T})$ is[8]

$$\{\tau_i,\rho_i,\sigma_k;\ i=1,\ldots,N;\ k=1,\ldots,N-1\}. \tag{1}$$

The generators $\{\tau_i\}$ and $\{\rho_i\}$ each take particle i along one of the fundamental noncontractable loops, depicted in Fig. 1(a), leaving all other particles fixed. We denote the subgroup of $B_N(\mathcal{T})$ generated by $\{\tau_i\}$ and $\{\rho_i\}$ by L_N. This is the so-called *unpermuted braid group*, which is the fundamental group for distinguishable particles. The generators $\{\sigma_k\}$ are the clockwise interchanges of particles k and $k+1$ for a configuration with no particles in the enclosed region [see Fig. 1(b)]. We denote by Σ_N the subgroup of $B_N(\mathcal{T})$ generated only by $\{\sigma_k\}$.

It is also convenient to introduce auxiliary generators $\{A_{ij}\}$ and $\{C_{ij}\}$ which move particle i around particle j along the paths in Fig. 1(c). These additional generators can be shown[8] to be related to the generators $\{\tau_i\}$ and $\{\rho_i\}$ by

$$A_{ij}=\rho_i^{-1}\tau_j\rho_i\tau_j^{-1},\quad C_{ij}=\tau_i^{-1}\rho_j\tau_i\rho_j^{-1}, \tag{2}$$

where $1\le i<j\le N$. We are always using a notation where the left-most operator is acting first.

The rather large set of relations[8] that defines $B_N(\mathcal{T})$

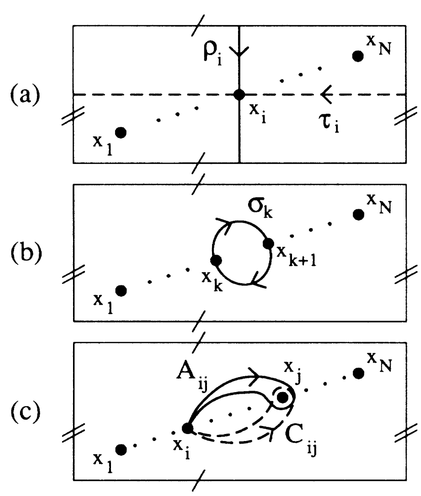

FIG. 1. Representative paths defining the generators of $B_N(\mathcal{T})$. The torus has been opened up and the generators are defined with respect to an ordered N-tuple of coordinates $\{x_1,\ldots,x_N\}$. (a) The generators $\{\tau_i\}$ and $\{\rho_i\}$, $1\le i\le N$. Note that the loops pass between the particles $\{1,\ldots,i-1\}$ and $\{i+1,\ldots,N\}$. (b) The local particle interchange generators $\{\sigma_k\}$, $1\le k\le N-1$. (c) The paths defining the auxiliary generators $\{A_{ij}\}$ and $\{C_{ij}\}$, $1\le i<j\le N$.

may be divided into three categories. The first category defines the unpermuted braid group L_N:

$$\tau_kA_{lm}=A_{lm}\tau_k,\quad \rho_kA_{lm}=A_{lm}\rho_k, \tag{3}$$

$$\tau_i\tau_j=\tau_j\tau_i,\quad \rho_i\rho_j=\rho_j\rho_i, \tag{4}$$

$$C_{ij}=(\tau_i^{-1}\tau_j^{-1})A_{ij}^{-1}(\tau_j\tau_i), \tag{5}$$

$$A_{ij}=(\rho_i^{-1}\rho_j^{-1})C_{ij}^{-1}(\rho_j\rho_i), \tag{6}$$

$$C_{ij}=(A_{j-1,j}\cdots A_{i+1,j})A_{ij}^{-1}(A_{i+1,j}^{-1}\cdots A_{j-1,j}^{-1}), \tag{7}$$

$$\rho_1^{-1}\tau_1^{-1}\rho_1\tau_1=A_{1,N}A_{1,N-1}\cdots A_{1,3}A_{1,2}, \tag{8}$$

where $1\le k<l<m\le N$ and $1\le i<j\le N$.

The second category involves only the generators $\{\sigma_k\}$:

$$\begin{aligned}&\sigma_k\sigma_l=\sigma_l\sigma_k,\quad 1\le k\le N-3,\quad |l-k|\ge 2,\\ &\sigma_k\sigma_{k+1}\sigma_k=\sigma_{k+1}\sigma_k\sigma_{k+1},\quad 1\le k\le N-2.\end{aligned} \tag{9}$$

Note that Eq. (9) gives $\sigma_k=\sigma_{k+1}\equiv\exp(i\theta)$ if all $\{\sigma_k\}$ commute, as they do for a 1D IUR. The angle θ is the *statistical angle*.

The third category mixes the generators of L_N and the generators of Σ_N:

$$\tau_{i+1}=\sigma_i^{-1}(\tau_i)\sigma_i^{-1},\quad \rho_{i+1}=\sigma_i(\rho_i)\sigma_i, \tag{10}$$

$$\tau_1\sigma_j=\sigma_j\tau_1,\quad \rho_1\sigma_j=\sigma_j\rho_1, \tag{11}$$

$$\sigma_i^2=A_{i,i+1}, \tag{12}$$

where $1\le i\le N-1$ and $2\le j\le N-1$.

For a 1D IUR, it follows from Eqs. (12) and (2) that $\exp(2i\theta)=1$ on a torus. Therefore, the scalar *theories* give only the ordinary Bose ($\theta=0$) and Fermi ($\theta=\pi$) statistics. Following Imbo, Imbo, and Sudarshan[5] the same conclusion may be made for all closed surfaces of higher genus by Abelianization of their braid-group presentations.[9]

Generalized fractional statistics.— Even for a system of N *distinguishable* free particles there are many distinct quantum theories corresponding to distinct IUR's of L_N. This ambiguity has nothing to do with the statistics, which describes the indistinguishability of identical particles. Therefore, following Ref. 5, we define the "statistics" provided by η, an IUR of $B_N(\mathcal{M})$, as $\eta\downarrow\Sigma_N$, the restriction of η to Σ_N. Two IUR's of $B_N(\mathcal{M})$, η_1 and η_2, we say are *statistically equivalent*[5] if $\eta_1\downarrow\Sigma_N$ and $\eta_2\downarrow\Sigma_N$ are isomorphic. On a multiply connected manifold $\mathcal{M}$ we can then consider M-dimensional IUR's η, which are statistically equivalent with scalar fractional-statistics theories,

$$\eta\downarrow\Sigma_N\cong\bar{\eta}\otimes 1_M, \tag{13}$$

where $\bar{\eta}$ is a 1D IUR of Σ_N, 1_M is the $M\times M$ unit matrix, $\otimes$ denotes tensor multiplication, and $\cong$ means equivalence as representations. In other words, the statistics (describing local interchange of identical particles) is scalar, whereas the quantum theory (which includes global paths) is nonscalar and non-Abelian. This construction is not realizable on simply connected manifolds $\mathcal{M}$, where $B_N(\mathcal{M})=\Sigma_N$. Hence it is possible to obtain fractional statistics in a new form we call *generalized fractional statistics.* The particles obeying this type of statistics are called *generalized anyons.* The generators σ_k now have the form $\exp(i\theta)\mathbf{1}_M$, whereas $\{\tau_i\}$ and $\{\rho_i\}$ are general unitary $M\times M$ matrices. The interpretation of such a representation is that the generators $\{\sigma_k\}$ act on M-component state vectors by multiplying all components by the *same* phase, $\exp(i\theta)$, whereas $\{\tau_i\}$ and $\{\rho_i\}$ mix the components. We may now ask which statistical angles θ are compatible with the complete set of rules determining the braid group.

Relations (10)-(12) give simple relations among $\{\tau_i\}$ and among $\{\rho_i\}$:

$$\tau_{i+1}=\tau_i e^{-2i\theta},\quad \rho_{i+1}=\rho_i e^{2i\theta}, \tag{14}$$

and together with Eq. (2) we have

$$A_{ij}=C_{ij}^{-1}=e^{2i\theta}\equiv A. \tag{15}$$

Finally, relations (3)-(8) together with Eq. (15) give only one more equation,

$$1=A^N=\exp(2Ni\theta). \tag{16}$$

This last equation is one of the conditions that determines the allowed values of θ. The other condition is a consequence of Eqs. (12), (15), (2), and (14). These imply that $\{\tau_i\}$ and $\{\rho_j\}$ must obey the commutation relation

$$\tau_i\rho_j=\rho_j\tau_i e^{2i\theta}. \tag{17}$$

By taking the determinant of both sides we immediately see that θ is restricted by

$$\exp(2Mi\theta)=1. \tag{18}$$

Therefore, all possible statistical angles θ are given by

$$\theta=(\pi/G)n,\quad 0\le n\le 2G-1, \tag{19}$$

where G is the greatest common divisor of M and N [$G=\mathrm{GCD}(M,N)$]. By using the braid-group presentations given in Ref. 9 one may see that for an orientable surface of genus $g>1$ the restriction on θ generally depends on both M, N, and g.

We may now try to find the most general form of τ_1 and ρ_1 satisfying Eq. (17). All other generators $\{\tau_i\}$ and $\{\rho_i\}$ are then determined through Eq. (14). Without loss of generality, τ_1 may be chosen to be a diagonal matrix, $\tau_1=\mathrm{diag}(z_1,\ldots,z_M)$. By explicit calculation it is then possible to show that all diagonal elements of ρ_1 must be zero and that the eigenvalues of τ_1 must fulfill the condition $z_i=cz_{P_i}$, where $P\in S_M$ is a permutation P: $(1,\ldots,M)\to(P_1,\ldots,P_M)$, and $c\equiv\exp(2i\theta)$. There are two cases.

(1) If c has no shorter period than M, i.e., $c^m\neq 1$ if $1\le m\le M$, then all $\{z_i\}$ must be different and ρ_1 must be a monomial matrix (a matrix with exactly one entry in each row and each column) corresponding to the permutation P with cycle length M. Since all permutations with the same cycle structure belong to the same conjugacy class, we can always transform τ_1 and ρ_1 to the generic form

$$\tau_1=e^{i\theta}\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & c & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & c^{M-1}\end{bmatrix},\quad \rho_1=e^{i\xi}\begin{bmatrix}0 & \cdots & 0 & 1\\ 1 & \cdots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & 1 & 0\end{bmatrix}, \tag{20}$$

where ϕ and ξ are real numbers and $c=\exp(2i\theta)$. It should be noted that $\tau_1^M\propto\mathbf{1}_M$ and $\rho_1^M\propto\mathbf{1}_M$, which imply a periodicity in the degenerate wave function given by M revolutions instead of one revolution around the torus. These representations are all irreducible.

(2) If c has a shorter period m such that $c^m=1$ and $1<m<M$, one can show that it is always possible to find a basis such that ρ_1 is a block-diagonal matrix, where each block is an $m\times m$ irreducible submatrix with the form given by Eq. (20). This means that these representations are all reducible.

We will now consider Haldane and Rezayi's generalization of the Laughlin wave function[10] for the fractional quantum Hall effect to a system with PBC.[11] This is a special case of a translationally invariant system with PBC in a homogeneous magnetic field which has a degenerate center-of-mass (c.m.) wave function.[12] In this system there is a set of m linearly independent solutions for every particle configuration and the state vector may be interpreted as an m-component wave function. Hence the necessary condition for having our type of fractional statistics is fulfilled.

The many-particle Laughlin wave function Ψ is constructed from one-particle states in the first Landau level. In the Landau gauge, $\mathbf{A}=-By\hat{\mathbf{x}}$, these one-particle wave functions have the form

$$\psi(x,y)=\exp(-y^2/2)f(z),\quad z=x+iy, \tag{21}$$

where $f(z)$ is an entire function. We consider a periodic lattice with basis vectors $\mathbf{L}_1=|L_1|$ and $\mathbf{L}_2=i|L_2|$. In order to fulfill periodic boundary conditions in the one-particle coordinates $\{z_i\}$, an integer number of flux quanta N_s must pass through the primitive cell spanned by $\mathbf{L}_1$ and $\mathbf{L}_2$. The analytic part of Ψ [the factor $\exp(-\sum y_i^2/2)$ is excluded], with N_e electrons and N_h quasiholes centered at $\tilde{z}_1,\ldots,\tilde{z}_{N_h}$, is[11]

$$F(\{z_i\};\{\tilde{z}_n\})=F^{\mathrm{c.m.}}(Z)\prod_{n=1}^{N_h}\prod_{i=1}^{N_e}\vartheta_1(\pi(z_i-\tilde{z}_n)/L_1\,|\,\tau)\times\prod_{i<j}\{\vartheta_1(\pi(z_i-z_j)/L_1\,|\,\tau)\}^m, \tag{22}$$

where

$$Z=\sum_{k=1}^{N_e} z_k+\frac{1}{m}\sum_{n=1}^{N_h} \tilde{z}_n . \tag{23}$$

Here $\tau=iL_2/L_1$ and $\vartheta_1(u\,|\,\tau)$ is the odd elliptic theta function which is entire and quasiperiodic in u. The c.m. wave function $F^{\text{c.m.}}(Z)$ is also a product of ϑ functions but has a degeneracy $m=(N_s-N_h)/N_e$. The important observation is that each quasihole coordinate $\tilde{z}_n$ is scaled by a factor $1/m$ in the c.m. coordinate Z. This means that even though the c.m. part $F^{\text{c.m.}}(Z)$ is quasiperiodic with periods $\mathbf{L}_\alpha$, it is quasiperiodic in $\tilde{z}_n$ with the longer periods $m\mathbf{L}_\alpha$. If we choose a correct basis, and translate one quasihole by L_α, it turns out that the degenerate states mix in exactly the way described by the explicit matrices of the braid-group generators τ_1 and ρ_1, with $M=m$, given by Eq. (20). This global behavior is only consistent with π/m statistics.

On the other hand, the action of σ_k may be simulated by moving two quasiholes around each other without changing the c.m. coordinate Z. Hence all degenerate parts of the wave function will change by the same phase $\exp(i\theta)$. For a system without PBC this phase has been evaluated both by considering the adiabatic phase change[13] and by rederiving the wave function as a multivalued wave function.[14] When considering the local action of σ_k and taking the thermodynamic limit at fixed m there are no essential differences between the systems with and without PBC. This means that $\theta=\pi/m$, also in our case.

Note also that the physical system should be invariant (except for a phase proportional to the fluxes inside the torus) under simultaneous translation of all coordinates $\{z_k,\tilde{z}_n\}$ by $\mathbf{L}_\alpha$. Hence it is necessary that N_h (the number of quasiholes) is a multiple of m. This is the same condition as that which must be satisfied in order to obtain π/m statistics.

These three facts imply that the quasiholes in the Laughlin wave function with PBC can be interpreted as *generalized anyons* with π/m statistics.

We have shown that it is possible to obtain fractional statistics, with $\theta=\pi p/q$ and $\mathrm{GCD}(p,q)=1$, on a torus, if the system is described by an M-component wave function, where M is a multiple of q. In addition, q must be a divisor of the number of generalized anyons N. Furthermore, we have derived a generic form for the unitary matrices representing the generators $\{\tau_i\}$ and $\{\rho_i\}$ in the irreducible case where $q=M$ and $\mathrm{GCD}(p,q)=1$, and we have seen that this form is applicable to the FQHE with PBC.

I am grateful to Stellan Östlund for encouraging and helping me to think through this subject. I have also benefited from enlightening discussions with H. Hansson, A. Karlhede, J. M. Leinaas, P. Elmfors, H. Johannesson, J. Brzezinski, B. S. Skagerstam, and S. Kivelson (communicated by S. Östlund). After this Letter was accepted similar results have been reported by Dagotto, Fradkin, and Wen.

[a]Electronic address: tfkte@fy.chalmers.se.

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