

Conservation Laws, Anisotropy, and “Self-Organized Criticality” in Noisy Nonequilibrium Systems

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It is argued in the context of noisy, nonequilibrium Langevin models that systems with conserving deterministic dynamics and noise which violates the conservation law always exhibit self-organized criticality—spatial and temporal correlations that decay algebraically under generic conditions. Systems with both conserving deterministic dynamics and conserving noise require spatial anisotropy to exhibit self-organized criticality.

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In two interesting recent papers, Hwa and Kardar¹ (HK) and Garrido *et al.*² (GLMS) studied classical, stochastic, nonequilibrium dynamical models which exhibit generic scale invariance. That is, the models have infinite correlation lengths, correlations therefore decaying only *algebraically* in space and time for generic parameter values. Such behavior stands in marked contrast to that of equilibrium systems, where scale invariance is typically not obtained generically, but only when a parameter (e.g., temperature) is tuned to a critical value. The occurrence of generic scale invariance is called “self-organized criticality” (SOC) by Bak, Tang, and Wiesenfeld,³ who make the intriguing suggestion that it underlies the common appearance in nature of both fractal structures⁴ and $1/f$ noise.⁵ (We reserve the term SOC for situations where the correlation length is infinite, so that not only temporal but spatial correlations are long ranged, since in the presence of conservation laws even equilibrium systems under generic conditions exhibit “long-time tails,”⁶ i.e., correlations that decay algebraically in time at a given point in space, even though spatial correlations decay exponentially. We also treat only spatially homogeneous systems, and only systems without a continuous symmetry, since frustrated systems such as spin glasses⁷ and systems with continuously broken symmetries can exhibit generic algebraic decays of spatial correlations even in equilibrium.)

The models of Refs. 1 and 2 respectively describe dissipative transport in anisotropic open systems such as sandpiles, and the diffusion of particles subject to an external field which produces a current in one particular direction. They are, therefore, very similar in spirit.⁸ Of particular significance is the fact that both sets of models have a conservation law, a feature which, coupled with the models’ nonequilibrium character, both groups of authors have identified as the ingredient essential to the emergence of SOC.

In this paper we try to elucidate and sharpen the conditions under which SOC can be expected to occur in a

large class of stochastic nonequilibrium systems. Our approach is to analyze, on the basis of symmetry and conservation laws, noisy, single-component field-theoretic models with relaxational dynamics, i.e., Langevin models; such models presumably represent coarse-grained versions of underlying microscopic systems of interest. Our main message is twofold: First, we argue that in “strictly conservative” systems—systems (such as those of Ref. 2 and certain of those of Ref. 1), wherein no allowed dynamical process can change the conserved quantity—a necessary and sufficient condition for the occurrence of SOC under generic conditions for spatial dimensionalities $d \geq 2$ in the presence of intrinsic *spatial anisotropy*, i.e., anisotropy that cannot be removed by rescaling. (The models of both Refs. 1 and 2 are anisotropic, and so constitute good examples of this principle.) For $d=1$, SOC is more difficult to attain; in fully conservative systems it does not occur even in the presence of spatial asymmetry (e.g., particles preferring to move to the left than to the right). For example, rather than exhibiting SOC, the model of GLMS has correlations that decay exponentially in space⁸ for $d=1$, which can thus be thought of as the “lower critical dimension”⁹ for this model. Second, we point out that when the conserved quantity is conserved “only on average,” i.e., the deterministic part of the Langevin equation of motion is strictly conservative, while the stochastic or noisy part allows occasional violations of the conservation law, one always obtains SOC, even in isotropic systems, and even for $d=1$.

It is interesting to note that the discrete “sandpile” models studied^{3,10} as paradigms of SOC, and some of the real sandpiles studied experimentally,¹¹ do combine conservative deterministic dynamics with nonconserving noise in the form of particles dropped randomly onto the pile. In these discrete systems, no new particles are dropped until the preceding avalanche has terminated: The noise thus has long-range correlations in time; this distinguishes these systems from the Langevin models

considered here, where the noise acts at all times, and is uncorrelated in time. Whether this difference changes the universality classes to which the systems belong is unclear, but it is noteworthy in and of itself that in conserving systems such as the Langevin models discussed below one obtains^{1,2} SOC without restricting the noise to act only between deterministic relaxation events such as avalanches.¹²

One can understand the main ideas underlying these conclusions by considering the simplest linear, noisy, Langevin model that conserves the spatial integral of some one-component field [$h(\mathbf{x}, t)$ in Ref. 1's notation]:

$$\partial h(\mathbf{x}, t)/\partial t = \nu_1 \nabla^2 h(\mathbf{x}, t) + \eta(\mathbf{x}, t). \quad (1)$$

Here ν_1 is a diffusion constant. The Gaussian random noise variable η can, as discussed in Ref. 1, be chosen either to conserve h strictly,

$$\langle \eta(\mathbf{x}, t) \eta(\mathbf{x}', t') \rangle = 2(D_\perp \nabla_\perp^2 + D_\parallel \nabla_\parallel^2) \delta(\mathbf{x} - \mathbf{x}') \delta(t - t'), \quad (1a)$$

or not to conserve it except on average,

$$\langle \eta(\mathbf{x}, t) \eta(\mathbf{x}', t') \rangle = 2D \delta(\mathbf{x} - \mathbf{x}') \delta(t - t'). \quad (1b)$$

We will refer to Eq. (1) with noise correlations (1a) and (1b), respectively, as models (1a) and (1b). In model (1a), we have allowed for the presence of spatial anisotropy by having two different constants, D_\perp and D_\parallel , describe the magnitude of the noise in two different subspaces (denoted \perp and \parallel), of the complete d -dimensional space on which $h(\mathbf{x}, t)$ is defined. The introduction of anisotropy in this way again follows Ref. 1 for simplicity; the conclusions would not change were we to divide the space into more than two inequivalent subspaces, etc. Note that there should, strictly speaking, be different diffusion constants for the \perp and \parallel directions in (1), but a trivial rescaling of space allows us to treat them as equal without loss of generality. Note, too, that one need not include terms such as $\nabla_i h(\mathbf{x}, t)$ in (1), even for anisotropic systems, such as those of Ref. 1, which lack reflection symmetry in some (say, the i th) direction. Such terms can be absorbed by the transformation $h(x_i, t) \rightarrow h'(x_i, t) \equiv h(x_i - t, t)$. [One should, however, allow for more general noise correlations in anisotropic, strictly conserving systems. Even restricting oneself, as we do here, to noise correlations that are purely local in space and time, one might encounter problems whose symmetry permits terms of the form $\nabla_\perp \nabla_\parallel \delta(\mathbf{x} - \mathbf{x}') \delta(t - t')$ in Eq. (1a). This possibility is discussed below.¹³]

Let us first consider model (1a), which is precisely the linear Langevin equation treated in Ref. 2. The model is, of course, exactly solvable, as is made explicit in that reference. In the anisotropic case $D_\perp \neq D_\parallel$, the model exhibits SOC, i.e., algebraic decays of the correlations of $h(\mathbf{x}, t)$ for arbitrary d and arbitrary parameter values.

This is readily seen from the expression for the k -dependent static susceptibility, $\chi(\mathbf{k}) \equiv \langle |h(\mathbf{k}, t)|^2 \rangle$, viz., $\chi(\mathbf{k}) = (D_\perp k_\perp^2 + D_\parallel k_\parallel^2) / \nu_1 |\mathbf{k}^2|$, whose Fourier transform, the (anisotropic) equal-time correlation function $G(\mathbf{x}) \equiv \langle h(\mathbf{x}, t) h(\mathbf{0}, t) \rangle$, decays with distance, x , like x^{-d} for large x . Note that, given our choice of spatial scaling, which sets the coefficients of ∇_\perp^2 and ∇_\parallel^2 equal in Eq. (1), the difference between D_\perp and D_\parallel cannot be eliminated by further rescaling of coordinates.

When $D_\perp = D_\parallel$, on the other hand, (1) reduces to the familiar time-dependent Ginzburg-Landau equation with conserved order parameter,⁶ associated with the Hamiltonian $H \sim \int h^2$. In other words, in the isotropic limit Eq. (1) satisfies detailed balance for this underlying Hamiltonian, the system is in equilibrium, and correlations decay *exponentially* in space (though algebraically in time) for generic parameter choices. [To see this easily from (1) requires the addition to the right-hand side of the higher-order gradient term $-\nu_2 (\nabla^2)^2 h(\mathbf{x}, t)$; one then finds spatial correlations decaying with correlation length $\xi \sim (\nu_2/\nu_1)^{1/2}$. In the nongeneric case wherein ν_2 and all coefficients of still higher powers of ∇^2 in (1) are taken to vanish, model (1a) with $D_\perp = D_\parallel$ gives $\chi(\mathbf{k}) = D_\perp / \nu_1$; i.e., $G(\mathbf{x}) = (D_\perp / \nu_1) \delta^d(\mathbf{x})$, so that $\xi = 0$.]

Since taking $D_\perp \neq D_\parallel$ in model (1a) both violates detailed balance and breaks spatial isotropy, there is some uncertainty as to whether in fully conservative systems the violation of detailed balance alone is sufficient to produce SOC. To see that typically it is not, and that spatial anisotropy is also required, consider a general, isotropic, fully conservative, local Langevin equation with all possible terms analytic in the field $h(\mathbf{x}, t)$ and its derivatives allowed. Such a model can be written¹⁴

$$\partial h(\mathbf{x}, t)/\partial t = \nu_1 \nabla^2 [f(\{h(\mathbf{x}, t)\})] + \eta(\mathbf{x}, t). \quad (2a)$$

Here the ∇^2 operator enforces conservation in the isotropic system, and

$$f(\{h\}) = h - (\nu_2/\nu_1) \nabla^2 h + \dots + u_1 h^2 + u_2 h^3 + \dots + w_1 (\nabla h)^2 + \dots, \quad (2b)$$

where u_1, u_2 , etc., are coupling constants; the noise correlations in (2a) satisfy

$$\langle \eta(\mathbf{x}, t) \eta(\mathbf{x}', t') \rangle = [D_1 \nabla^2 + D_2 (\nabla^2)^2 + \dots] \times \delta(\mathbf{x} - \mathbf{x}') \delta(t - t').$$

Consider the nonlinear terms of (2b): The simple powers of h can be derived from a Hamiltonian. That is, they satisfy detailed balance in the long-wavelength limit, and so cannot be expected to produce SOC. The lowest-order term which violates the detailed balance condition is¹⁵ the w_1 term. It is straightforward to verify by renormalization-group (RG) methods^{6,15} that in any dimension this term is, at least for small w_1 , an *irrelevant* perturbation, i.e., that it cannot change the

long-distance, long-time behavior of correlation functions given by the linear terms of (2b). Indeed, all of the nonlinear terms in (2b) are irrelevant, whether or not they can be derived from Hamiltonians. The same is true of the still higher terms indicated by the ellipsis in (2b). Thus the short-range (exponentially decaying) correlations predicted by the linear terms of model (2) [i.e., of model (1a) with $D_{\perp}=D_{\parallel}$] are robust against the addition of small nonlinearities. This seems very natural physically: It would be surprising to have the extra fluctuations represented by the nonlinear terms of (2) *increase* the range of correlations present in the linear theory. We conclude that (barring the occurrence of a stable strong-coupling fixed point that gives rise to SOC for *large* values of the nonlinear coupling constants¹⁶), isotropic, fully conserving systems of the type (2) do *not* exhibit SOC, even when they violate detailed balance.

It follows that the SOC derived in Ref. 2 for model (1a) with $D_{\perp}\neq D_{\parallel}$ is a consequence not merely of the model's nonequilibrium character, but of its spatial anisotropy. [Note that even if $D_{\perp}=D_{\parallel}$, higher-order anisotropic terms like $\nabla_{\parallel}^4\delta(\mathbf{x}-\mathbf{x}')$ on the right-hand side of (1a) are sufficient to produce SOC.] It remains to be seen whether the generic scale invariance of this linear anisotropy theory survives the inclusion of nonlinear terms. As pointed out in Ref. 1, the lowest-order nonlinearity consistent with the conservation law that can be added to the right-hand side of Eq. (1) is $\nabla_i[h(\mathbf{x},t)]^2$. This term is obviously appropriate only for systems without reflection symmetry in the i th direction. Simple power counting^{1,6} shows that it is irrelevant for any dimension d greater than the upper critical dimension, $d_u=2$. All nonlinear terms containing either higher gradients or more powers of the field h than this lowest nonlinearity can be shown irrelevant for all $d>1$. Now for $d=1$, even the linear (strictly conserving) theory (1a) has exponentially decaying correlations rather than SOC, since there are not enough directions available to have $D_{\perp}\neq D_{\parallel}$. It is unreasonable to expect that nonlinear fluctuations will *increase* the range of correlations; one concludes⁸ that no SOC can occur for $d=1$ in the strictly conserving case. Restricting oneself to integer dimensions, therefore, one is left with only $d=2$ and nonlinearities of the form $\nabla_i h^2$ to consider.

HK (see also Ref. 8) have already treated the 2D case with reflection symmetry missing in only one of the two directions, i.e., model (1a) wherein the \parallel and \perp subspaces comprise one dimension each, and with the term $\nabla_{\parallel}h^2$ added to the right-hand side. They show by RG methods that in this case the nonlinearity is marginally irrelevant, so that SOC persists. That leaves only the maximally asymmetric possibility, with both $\nabla_{\parallel}h^2$ and $\nabla_{\perp}h^2$ added to the right-hand side of (1), their coefficients being different in the absence of any special symmetry that dictates otherwise. [It turns out that in this situation symmetry also admits the contribution $\nabla_{\perp}\nabla_{\parallel}\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')$ to the noise correlation of Eq.

(1a).] It is straightforward to verify by RG analysis (details to be given elsewhere⁸) that both the nonlinear terms are likewise marginally irrelevant in this case [and that, as in Ref. 8(b), the system satisfies detailed balance right at the fixed point]. Thus SOC is always obtained in strictly conserving, anisotropic situations.

Let us turn now to model (1b), where the noise does not conserve the field $h(\mathbf{x},t)$ strictly, but only on average. Thus the deterministic part of the equation is conserving but the noise is not, a mismatch which breaks detailed balance,^{6,14} making the model a nonequilibrium one. It is easy to check by explicit solution that under these conditions the linear theory (1) gives rise to SOC whether or not there is any spatial anisotropy in the system: The static susceptibility $\chi(\mathbf{k})$ diverges like $1/k^2$ for small k , producing equal-time spatial correlations $G(\mathbf{x})$ that decay like $x^{-(d-2)}$. To show that SOC survives the addition of nonlinearities, one need only make the (rather mild) assumption that $\chi(\mathbf{k})$ is a continuous function of \mathbf{k} near $k=0$. This assumption suffices to demonstrate that SOC persists in the presence of arbitrary nonlinear terms on the right-hand side of (1), provided only that those terms conserve the field h . The point is that conservation implies the equation

$$\partial h(\mathbf{k}=0,t)/\partial t = \eta(\mathbf{k}=0,t) \quad (3)$$

for the $\mathbf{k}=0$ Fourier component of h . Equation (3), together with the noise correlations (1b), imply that $h(\mathbf{k}=0,t)$ undergoes a random walk in time. Thus $\langle |h(\mathbf{k}=0,t)|^2 \rangle$ diverges like t as $t \rightarrow \infty$; i.e., the static susceptibility $\chi(\mathbf{k}=0)$ is infinite. The continuity assumption then implies that $\chi(\mathbf{k})$ blows up as $k \rightarrow 0$, implying long-ranged, rather than exponentially decaying, equal-time spatial correlations. The most likely scenario is that, as in the linear theory, these correlations decay *algebraically* in space, though any decay slow enough to produce an infinite susceptibility is conceivable. The model of Ref. 1, wherein the nonlinear operator $\nabla_{\parallel}h^2$ is added to model (1b) (with one particular direction designated as \parallel), is an example of a nonlinear theory with algebraic decays for all d , the linear theory being quantitatively correct above the upper critical dimension of 4.

Thus we expect that the peculiar (manifestly nonequilibrium) combination of conserving deterministic dynamics and nonconserving noise produces SOC quite generally, both for isotropic and anisotropic systems. Unlike for the fully conserving models considered earlier, there seems no reason not to expect SOC to obtain in dimensions down to and including $d=1$ in this case.¹² It is worth reiterating that, while both discrete sandpile models^{3,10} and real experimental sandpiles¹¹ do combine conservative deterministic dynamics with nonconserving noise in the form of particles dropped randomly on the pile, the noise acts only between individual avalanches, i.e., has rather special correlations.

Several systems without conservation laws either ex-

hibit or have been suggested to exhibit SOC. Perhaps the best known of these are ordinary continuum interface models^{15,17} with translational symmetry, i.e., wherein the interface can be uniformly translated in a direction transverse to itself without any energy cost. The spatial correlations in such systems are indeed algebraic¹⁷ under generic conditions. It may, however, be somewhat misleading to describe these systems as having SOC, since translational invariance of the interface does not occur generically, but only when the parameter (e.g., gravity), which tends to pin the interface at a particular height, is set to zero. (One might argue that the imposition of a conservation law is likewise nongeneric and amounts to adjusting a parameter, so precisely what one calls SOC becomes a matter of taste; conservation laws are, however, ubiquitous.)

Models for the spread of forest fires or disease¹⁸ have also been proposed as displaying SOC, but these quite explicitly require the adjustment of a parameter to achieve power-law decay of correlations, and so do not seem fundamentally different from equilibrium systems wherein algebraic decays are obtained by tuning, e.g., the temperature, to a critical value. Interesting recent suggestions¹⁹ that the "game of life" cellular automaton exhibits SOC remain as yet imperfectly understood.

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