Wave Chaos in Singular Quantum Billiard

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We present a solvable singular quantum billiard which displays fully developed wave chaos. Its level statistics is investigated and proved to coincide with predictions of the Gaussian orthogonal ensemble of random matrices. The corresponding wave functions are shown to be well approximated by a Gaussian random variable.

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The statistical properties of quantum levels have attracted much interest during the last few years in connection with the speculation that it may reflect the degree of order in the corresponding classical system. It has been conjectured that the classically integrable systems lead to uncorrelated quantum levels (Poisson distribution), while the eigenvalues of systems which are classically chaotic were assumed to have the same statistical properties as the eigenvalues of random matrices belonging to the Gaussian orthogonal ensemble (GOE) (Wigner distribution). This fact has been confirmed in a number of numerical studies performed on various model Hamiltonians.[1-4] Berry and Tabor[5,6] showed that this behavior of quantum levels is a consequence of the semiclassical analysis and is indeed valid in a generic situation.

It is, however, well known that certain systems violate the above-described correspondence: There are integrable systems which do not have a Poisson distribution (for instance, a particle in a square well). Even worse, recent numerical studies performed on pseudointegrable billiards showed that one can find GOE statistics even in systems which have nonchaotic classical physics.[7]

These numerical results, though illustrative, cannot, however, serve as proof that the reversed correspondence (i.e., GOE statistics for classically nonchaotic systems) indeed appears. The point is that the numerical data are available only for the first N levels ($N \leq 400$ in a typical situation). It can happen that the spectral statistics change for high enough energies and become Poissonian and hence in agreement with the folk wisdom. This was also the way in which Cheon and Cohen explained their finding.[7] They argued that the GOE statistics change to Poissonian when approaching the semiclassical regime. On the other hand, the system studied in Ref. 7 consists of a square billiard with a number of rectangular pieces removed. But the presence of convex rectangular corners makes the semiclassical limit of this system questionable. The smallest relevant scale is represented by the edges of the corners (which are in fact point objects) and the semiclassical regime is expected to set in only for $E \to \infty$. One can therefore argue that the quantum waves will be strongly influenced by the edges and, consequently, that the GOE statistics will be present for all energies.

It is therefore interesting to see whether one can find a system where the influence of the measure-zero objects like sharp edges can be investigated and where the reversed statistics can be *proved* to appear.

The motivation for constructing such a model comes from the Sinai billiard where a point particle moves inside a rectangle well with a circular specularly reflecting obstacle with radius R. Sinai proved[8] that this system is classically fully chaotic for all $R > 0$. The corresponding quantum system has been investigated by Berry[9] and Bohigas, Giannoni, and Schmidt[10] who demonstrated numerically that its level statistics coincide with the GOE predictions. In such a way the Sinai billiard served as a good prototype for the right connection between the Hamiltonian of a quantized chaotic system and the GOE.

Our argument goes as follows: We start with the Sinai billiard and shrink the radius R of the circular obstacle to zero, replacing it by a point scatterer. It is not difficult to see that the classical system we obtain in such a way is not chaotic. The point is that the classical trajectories which are influenced by the scatterer are of measure zero (these are simply the trajectories which hit the scattering point) and cannot therefore rise the Kolmogorov entropy. In this sense the classical system does not "feel" the point scatter.

The quantum system behaves, however, differently: The quantum waves are substantially influenced by the point scatterer. One can therefore assume that the presence of the scatterer will eventually develop a wave chaos similar to this in the quantized Sinai billiard with $R > 0$.

Summarizing the above heuristic arguments on the $R \to 0$ limit we get the following: The classical system ceases to be influenced by the point scatterer and becomes nonchaotic while the quantum wave chaos is expected to survive and, as a result, the reversed level statistic is expected to appear.

We restrict ourselves now to the quantum case and proceed to precise formulations. The first step in this direction is to construct the corresponding quantum Hamiltonian. The difficult point is, of course, the description of the point scatterer. In order to handle it we will employ the theory of singular interactions.[11] The relevant operator is then constructed as follows: We start with the standard Hamiltonian H of the rectangular billiard Ω,

$$H = -\Delta, \qquad (1)$$
$$D(H) = \{ f \in L^2(\Omega);\ f = 0 \text{ on } \delta\Omega \},$$

where Ω is the rectangle

$$\Omega = \left[0, \frac{\pi}{a} \right] \times [0,\pi]$$

with a being a positive irrational number (we take $a = \sqrt{5} - 1$ in all numerical calculations). We remove now the relevant scattering point $(x_0, y_0) \in \Omega$ restricing H to H_0,

$$H_0 = H \mid D_0, \qquad (2)$$

with

$$D_0 = \{ f \in D(H);\ f = 0 \text{ in some neighborhood of the point } (x_0, y_0) \}.$$

What we get in such a way is a symmetric operator which is, however, not self-adjoint. In order to get the desired Hamiltonian we have to specify what is going on when the particle hits the point (x_0, y_0). This question is formally solved by constructing the self-adjoint extensions of H_0. At this point the self-adjoint extension theory developed by von Neumann can be applied:[12] We finally get a one-parameter family of Hamiltonians H_α,

$$H_\alpha = -\Delta, \qquad (3)$$
$$D(H_\alpha) = \{ f \in L^2(\Omega);\ f = 0 \text{ on } \delta\Omega,\ L_0(f) = \alpha L_1(f) \},$$

with L_0, L_1 defined as

$$L_0(f) = \lim_{\rho \to 0} \frac{f(x,y)}{\ln\rho},$$
$$L_1(f) = \lim_{\rho \to 0} [f(x,y) - L_0(f)\ln\rho], \qquad (4)$$

and

$$\rho = [(x - x_0)^2 + (y - y_0)^2]^{1/2}.$$

The parameter α, $\alpha \in (-\infty, \infty)$, can be interpreted as the coupling constant of the point scatterer placed at (x_0, y_0). (One can also obtain the operator H_α by directly performing the limit $R \to 0$; see Ref. 13 for details.)

The wave dynamics governed by H_α is most easily understood in an analogous electromagnetic model, where it describes the transverse modes of an electromagnetic field propagating inside a rectangular waveguide with perfectly conducting walls (Dirichlet boundary conditions on $\delta\Omega$). In this case the point scatterer is realizable by a straight charged wire which is placed inside the waveguide parallel to its walls. The coupling constant α is then connected with the charge of the wire.

Let us investigate the spectrum of H_α. To compute the eigenvalues directly would be, however, a difficult task. We will therefore employ the self-adjoint extension theory and calculate the corresponding resolvent:

$$(H_\alpha - z)^{-1} = (H - z)^{-1} - \frac{2\pi a}{1 + 2\alpha\xi(z)} | g_z(x,y,x_0,y_0) \rangle \times \langle g_z(x_0,y_0,x,y) |, \qquad (5)$$

where $g_z(x,y,x',y')$ is the Green's function of H,

$$[(H - z)^{-1} f](x,y) = \int_\Omega g_z(x,y,x',y') f(x',y') dx'dy', \qquad (6)$$

and $\xi(z)$ is a meromorphic function given by

$$\xi(z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{4a \sin^2(nax_0)\sin^2(my_0)}{\pi(n^2a^2 + m^2 - z)} - \frac{1}{2m} \right). \qquad (7)$$

The spectrum of H_α now can be obtained by investigating the pole structure of the resolvent: It is determined by the transcendental equation

$$1 + 2\alpha\xi(z) = 0. \qquad (8)$$

The eigenvalues are not degenerate and the wave functions can be easily found from the residuum of the corresponding resolvent pole. This calculation leads to

$$f_n(x,y) = \frac{4a}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin(nax)\sin(my)\sin(nax_0)\sin(my_0)}{n^2a^2 + m^2 - E_n}, \qquad (9)$$

where E_n is the solution of (8).

Our main aim is to prove that the level-spacing distribution $P(s)$ of H_α obeys the predictions of the GOE, i.e., that

$$P(s) \approx ks \qquad (10)$$

for small s and all $\alpha \neq 0$. (k is a positive constant depending on α.) For the sake of simplicity we will sketch here the proof only for $\alpha = \infty$ and postpone the details for a subsequent publication. [It is worth noting that the spacing distribution also depends on the number-theoretical nature of the scatterer position (x_0, y_0).]

In the case $\alpha = \infty$ the spectrum of H_α is determined by the *zeros* of the function $\xi(z)$ while the spectrum of the "free" billiard Hamiltonian H is given by the *poles* of ξ. We know, however, that the spectrum of H (which is given by eigenvalues $a^2n^2 + m^2$, $n,m = 1,2,\ldots$) possesses a spacing distribution which is fairly close to Poisson.[14] Thus the question about the level statistics for H_α can be reformulated as follows: *What is the spacing statistics for zeros of a meromorphic function of the form (6), the poles of which are determined by a Poisson process*? The answer is readily obtained: A cluster of roots can occur only if there is a corresponding cluster of poles. Let us suppose that we have a cluster which contains two

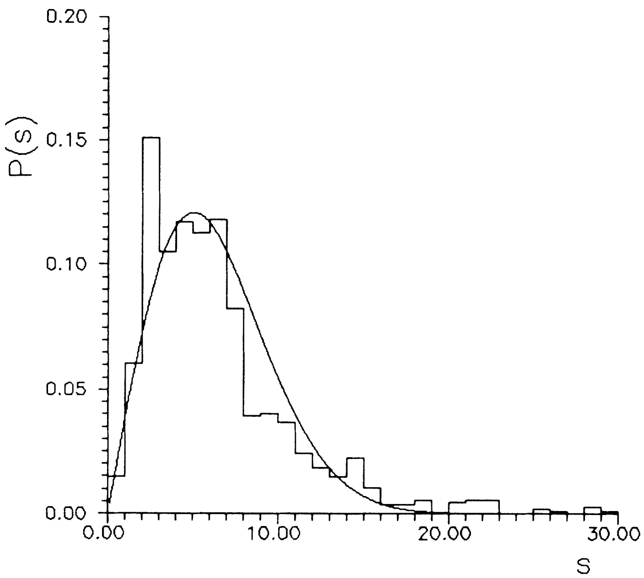

FIG. 1. The level-spacing distribution of the Hamiltonian H_α calculated for $\alpha=1$ is compared with the predictions of the GOE (Wigner distribution). The scatterer has been placed in the center of the billiard and only the states with even parity have been taken into account.

poles; i.e., we find exactly two poles within an interval of length s with $s \to 0$. (The probability of finding such a cluster tends to 1 for $s \to 0$ since the poles are Poisson distributed.) It is clear that in this case there will be exactly one zero of ξ localized within this interval. On the other hand, a cluster of zeros appears inevitably as soon as three or more poles cluster together ($n-1$ zeros within each cluster of n poles). The probability that this happens is, however, determined by the *convolution* of the Poisson distribution. The leading term comes from the three pole clusters and leads directly to (10). As a result we find a Wigner-like level repulsion in a situation where the Poisson behavior is to be expected (the classical system corresponding to H_α is not chaotic). The spacing distribution calculated for $\alpha=1$ is plotted in Fig. 1 and compared with the GOE predictions.

Is this a surprising result? It has been argued[7] that the relationship between the quantum level statistics and the classical motion takes place only in the semiclassical regime. The semiclassical regime sets in, however, only with the typical wavelength being much shorter than all the relevant scales of the problem. In our case this happens only for $E \to \infty$ since the relevant scale is determined by the point scatterer. The quantum mechanics remains in such a way out of the semiclassical regime for all energies and the standard relationship between the level statistics and classical dynamics breaks down. However, one has to be careful with the semiclassical arguments since the limits $R \to 0$ and $\hbar \to 0$ do not commute.

The analogy of our model with the typical chaotic bil-

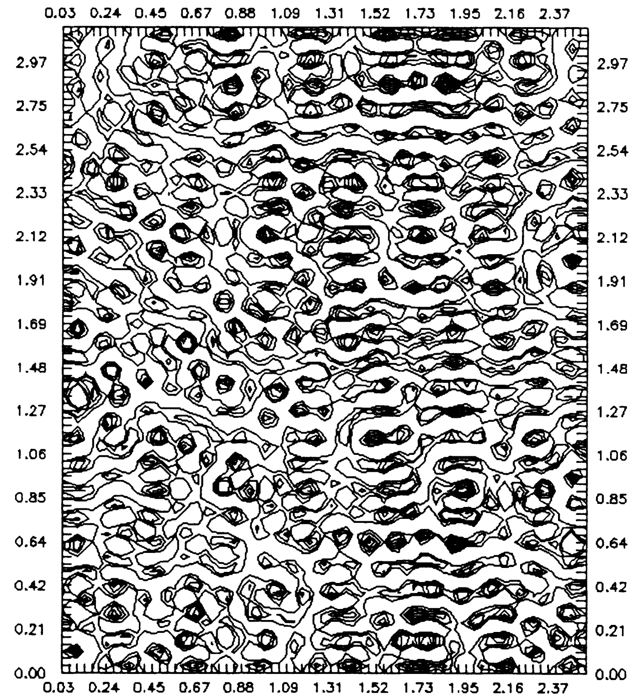

FIG. 2. The topography of the positive part of the eigenfunction corresponding to the 411 eigenvalue of H_α with $\alpha=100$. The point scatterer is placed at the point ($0.55\pi/a$, 0.65π).

liard goes, however, beyond the spectral statistics. Also, the corresponding wave functions display typical chaotic structures[15,16] and become messy irregular-looking objects (see Fig. 2).

The description of the wave function in a chaotic quantum billiard is based on the concept of the eikonal theory, according to which the wave function ψ can be written as a sum over eikonal wavelets which propagate along the classical trajectory. In a chaotic billiard the trajectory is chaotic and ψ is assumed to be of the form

$$\psi(x,y) \approx \sum_j \exp\{i\sqrt{E}\,[\sin(\omega_j)x+\cos(\omega_j)y+\phi_j]\}\,, \tag{11}$$

where ω_j and ϕ_j are independent random variables. It follows from this assumption that the wave function ψ of the quantized classically chaotic billiard is a Gaussian random variable. The probability of finding a value of ψ at any point is thus given by[16]

$$p(\psi)=\frac{1}{(2\pi)^{1/2}\sigma}e^{-\psi^2/2\sigma^2} \tag{12}$$

with $\sigma=1/\sqrt{S}$, where S is the area of the billiard.

It is simple to check whether this assumption also works for our model. Using the formula (9) we evaluated the wave function corresponding to the 411th state

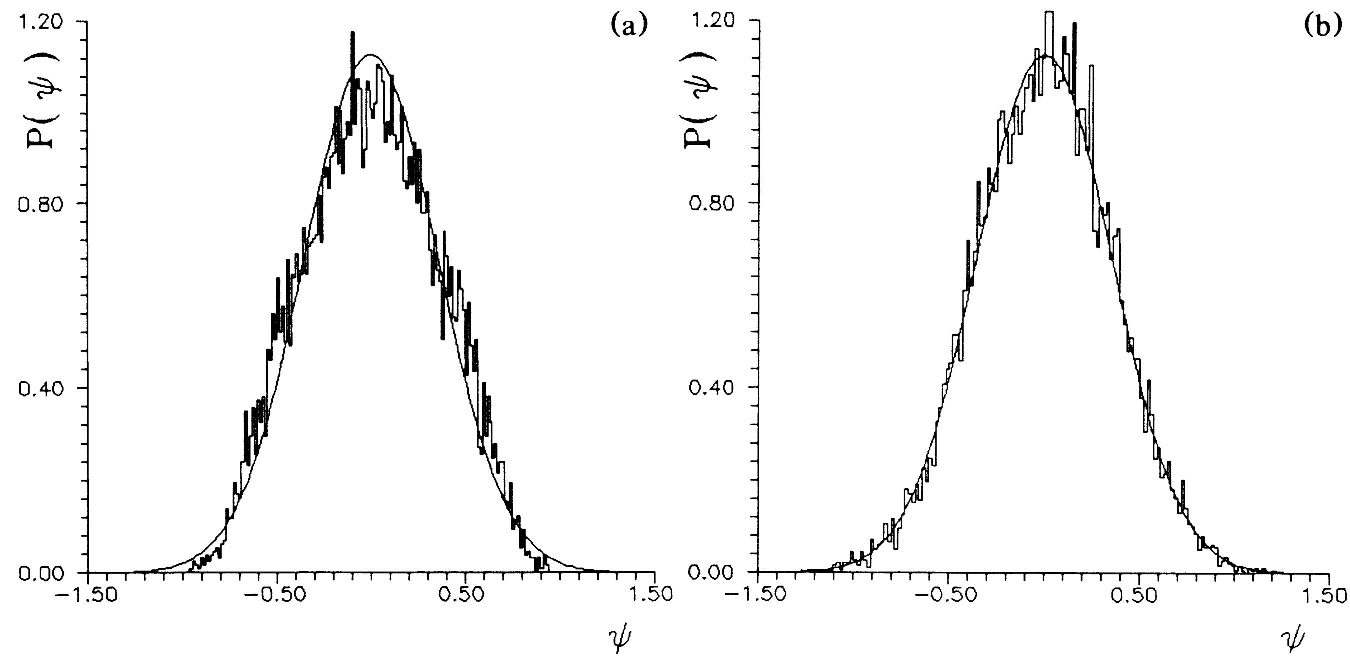

FIG. 3. (a) The probability distribution of the values of the eigenfunction ψ compared with the predictions obtained for a Gaussian random variable. (b) The probability distribution for a function ψ obtained as a random superposition of 500 plane waves propagating in random directions with random phases compared with the normal distribution (12).

of H_α with $\alpha=100$ at 10000 points and constructed the probability distribution as a normalized histogram with 150 bins. [The position of the scatterer is $(x_0,y_0)=(0.55\pi/a,0.65\pi)$.] The result is plotted in Fig. 3(a) and compared with the expected distribution (12) with $\sigma=\sqrt{a}/\pi$. A similar calculation has also been done for the random wavelet superposition (11) with 500 terms and the result is plotted in Fig. 3(b).

In conclusion, it appears that one can find a fully developed wave chaos also in systems which are classically not chaotic. Such systems are expected to contain parts (sharp edges, scattering points, etc.) which do not influence the classical dynamics (they are of measure zero) but which nevertheless lead to strong wave scattering.

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[1]B. Eckhardt, Phys. Rep. **163**, 207 (1988).

[2]R. Aurich, M. Sieber, and F. Steiner, Phys. Rev. Lett. **61**, 483 (1988).

[3]E. Caurier and B. Grammaticos, Phys. Lett. A **136**, 387 (1989).

[4]K. M. Christofel and P. Brumer, Phys. Rev. A **33**, 1309 (1986).

[5]M. V. Berry and T. Tabor, Proc. Roy. Soc. London A **356**, 375 (1977).

[6]M. V. Berry, Proc. Roy. Soc. London A **400**, 229 (1985).

[7]T. Cheon and T. D. Cohen, Phys. Rev. Lett. **62**, 2769 (1989).

[8]Ya. G. Sinai, Russian Math. Surveys **25**, 137 (1970).

[9]M. V. Berry, Ann. Phys. (N.Y.) **131**, 163 (1981).

[10]O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).

[11]S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics* (Springer-Verlag, Berlin, 1988).

[12]M. Reed and B. Simon, *Modern Methods of Mathematical Physics* (Academic, New York, 1978), Vol. 2.

[13]J. P. Antoine, F. Gesztesy, and J. Shabani, J. Phys. A **20**, 3587 (1987).

[14]G. Casati, B. V. Chirikov, and I. Guarneri, Phys. Rev. Lett. **54**, 1350 (1985).

[15]E. J. Heller, Phys. Rev. Lett. **53**, 1515 (1984).

[16]S. W. McDonald and A. N. Kaufman, Phys. Rev. A **37**, 3067 (1988).