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# Scaling Properties of Band Random Matrices

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It is shown on the basis of numerical data that the normalized localization length of eigenvectors of band random matrices follows a scaling law. The scaling parameter is $b^2/N$, where $b$ measures the bandwidth and $N$ is the size of the matrix.

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In a previous investigation[1,2] on the statistical properties of quantum "chaotic" systems, it was found that in the particular model of the kicked rotator on the torus, localization effects display a scaling behavior. This scaling is connected with the quantum suppression of classical dynamical chaos[3,4] and has a counterpart in the scaling theory of localization for one-dimensional disordered systems of finite size.[5]

The one-period evolution of the kicked rotator, in the angular momentum representation, is given by a unitary $N\times N$ matrix. The matrix elements are appreciably different from zero only inside a band of size $2k$, where $k$ is the strength of the perturbation. Outside the band they decay exponentially. In the case of classical strong chaos, the matrix elements can be considered as pseudorandom numbers, and when $k$ is large enough, the unitary matrix exhibits the statistical properties of the circular-orthogonal ensemble.[6] The scaling parameter which describes the statistical properties in the regime of full classical chaos is the ratio $k^2/N$. The quantity $k^2$ is proportional to $\xi_\infty$, the localization length measured through the rate of exponential decay of the eigenvectors in the limit of infinite size ($N\rightarrow\infty$).

The natural question then arises whether scaling behavior is a general property of random matrices with a band structure. This is an interesting mathematical problem, which is also relevant for physics. Indeed, band random matrices may be regarded as models for quantum systems whose states are only partially coupled to each other by the interaction. For example, this feature is common to many models of solid-state physics[7] and may be relevant for several problems in atomic and nuclear physics.[8]

In this paper, we consider ensembles of real symmetric band random matrices (BRM). The statistical properties of such matrices are poorly understood, except in the limiting cases of Gaussian-orthogonal-ensemble (GOE) (Refs. 9–11) and tridiagonal matrices. In the latter case, the eigenvectors display an exponential localization in the large-$N$ limit: $\psi_n \approx \exp(-|n-n_0|/\xi)$ (Ref. 7), where $\xi$ is the inverse of the Lyapounov exponent, computed, for example, by means of Thouless's formula[12] or by the transfer-matrix method.[13] The investigation of the intermediate situation, with particular reference to the localization properties of eigenvectors, is the object of this paper. Unlike the random-matrix theory, for which many analytical treatments are available, the lack of rotational invariance of BRM ensembles makes the use of computer simulation unavoidable at this stage.

Before presenting our results, we would like to mention that a particular class of band matrices ("bordered matrices") has been considered by Wigner.[14] They are characterized by integer diagonal entries $\ldots,-2,-1,0,1,\ldots$ and a band of size $b$ of matrix elements $a_{ij}=\pm h$, where $h$ is constant and the sign is random; outside the band $a_{ij}=0$. The model is analytically solvable in the tridiagonal case, and exhibits a semicircle distribution of the eigenvalues in the limit $b$ and $h\gg 1$, with

$h^2/b$ finite. Bandlike random matrices have also been considered in Refs. 15 and 16.

The BRM ensemble is defined as the set of real symmetric $N \times N$ matrices with elements $a_{ij}=0$ for $|i-j| \geq b$. Therefore, $b$ is the number of nonzero elements in the first row and equals 1 for diagonal, 2 for tridiagonal, and $N$ for GOE matrices. The matrices being symmetric, the number of independent matrix variables is given by $F=b(2N-b+1)/2$. They are chosen as independent Gaussian random numbers with mean equal to zero and variances such that, up to a normalization constant, the probability density for one matrix $A$ in the ensemble is

$$\mathcal{P}(A)=e^{-\omega \operatorname{Tr} A^2}=\prod_{i=1}^{N} e^{-\omega a_{ii}^2} \prod_{i<j} e^{-2\omega a_{ij}^2}. \qquad (1)$$

From expression (1) one easily computes the ensemble $\langle \operatorname{Tr} A^2 \rangle = F/2\omega$.

The ensemble is fully characterized by the parameters $N$, $b$, and $\omega$. The last one sets the scale of eigenvalues and plays no role in the statistics of spacings nor in the values of the eigenvectors (a change in $\omega$ amounts to a multiplication of the matrix by a constant factor, which does not affect the eigenvectors). We take advantage of this fact to scale the eigenvalue in such a way that $\langle \operatorname{Tr} A^2 \rangle = N$.

As an interesting preliminary result of our investigations, we have found that the eigenvalues are distributed according to an approximate semicircle law with radius 2, as would be for the GOE in the large-$N$ limit. The outcome of a semicircle distribution is not obvious, although seems to be a general feature of random matrices.[17]

In the following we provide the numerical evidence for the existence of a scaling property for the localization of eigenvectors. The basic scaling variable is

$$x=b^2/N \qquad (2)$$

in analogy to the kicked-rotator model ($b \approx k$). In Figs. 1(a) and 1(b) we show two typical examples of the structure of chaotic states of band random matrices in the two extreme cases $x \ll 1$ and $x \gg 1$. They look completely different: The former eigenvector [Fig. 1(a)] shows an exponential decay, while the latter [Fig. 1(b)] apparently fills in a random way the whole available length; a consistent definition of localization for both cases is not straightforward.

In order to introduce a measure for the localization of chaotic eigenvectors we follow the same approach described in Refs. 1 and 18. For each normalized eigenvector $(u_1, \ldots, u_n)$ we introduce the information entropy

$$H(u_1, \ldots, u_N) = -\sum_{i=1}^{N} u_i^2 \ln u_i^2, \quad \sum_{i=1}^{N} u_i^2 = 1. \qquad (3)$$

The quantity $\exp(H)$ is proportional to the effective number of nonzero components of the eigenvector and

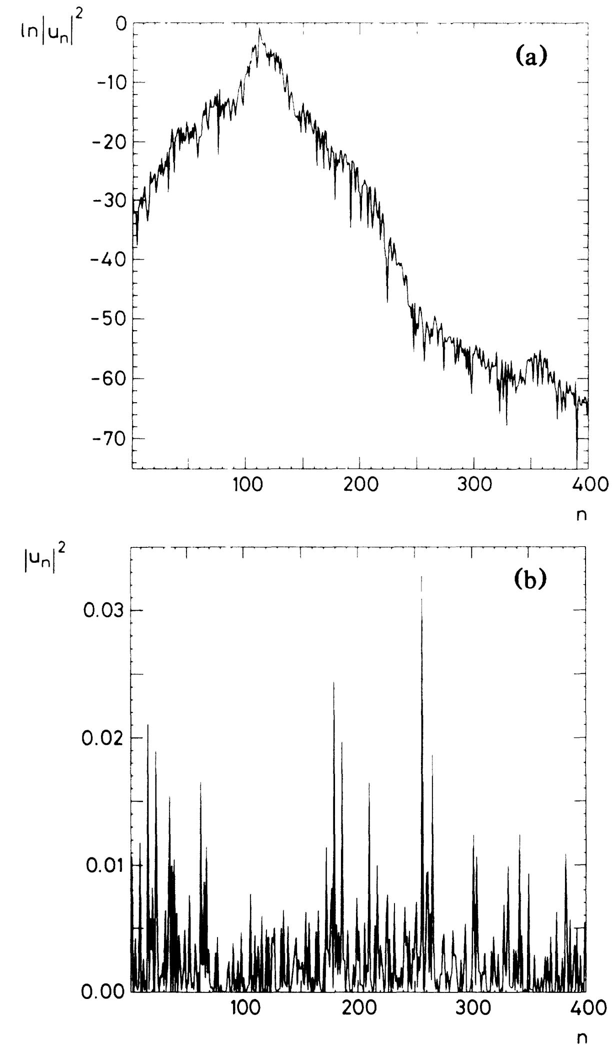


FIG. 1. Squared components $u_n^2$ vs $n$ for typical BRM eigenvectors, with $N=400$. (a) $b=4$, a typical exponentially localized state (logarithmic scale). (b) $b=50$, a typical delocalized chaotic state (ordinary scale).

therefore gives a measure of localization. It has been used, for example, in the investigation of solid-state models,[19] and of the quantum dynamics of classically chaotic systems.[20,21] In the present paper, following Ref. 18, we introduce a normalizing factor in order to obtain a definite quantity with values between 0 and $N$, $N$ being the size of the matrix. This factor is crucial for the purpose of our paper since it takes into account the chaotic structure of the states. To this end, let us consider the limit case of the GOE, with completely delocalized and chaotic states. It is well known that GOE states, as a consequence of the $O(N)$ invariance of the ensemble, have a uniform distribution over the surface of the $N$-sphere of radius 1, with a probability density for each component given by (Ref. 11)

$$w(u_k)=\frac{\Gamma(N/2)}{\Gamma(N/2-\frac{1}{2})\Gamma(\frac{1}{2})}(1-u_k^2)^{(N-3)/2}. \qquad (4)$$

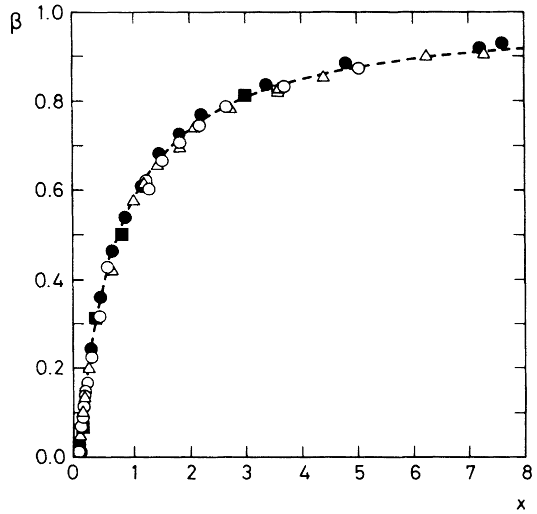


FIG. 2. The scaled localization length $\beta$ vs $x=b^2/N$ for $N=200$ (●), $N=400$ (△), $N=600$ (○), $N=800$ (■), and $N=1000$ (□). Numerical data show a remarkable scaling behavior with the scaling parameter $x$. The dashed curve is given by expression (9) and fits quite satisfactorily the numerical data.

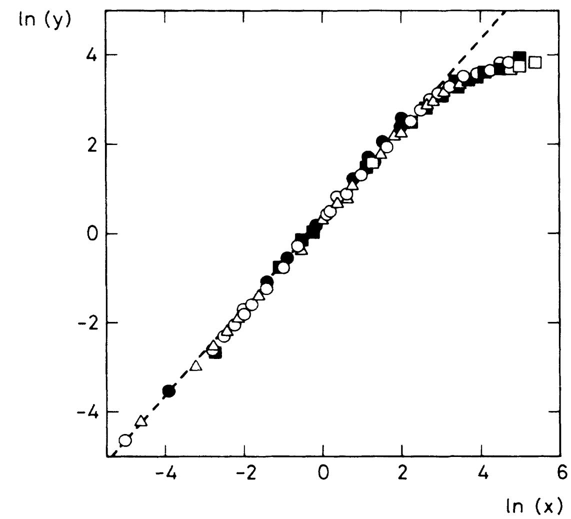


FIG. 3. A log-log plot of the data of Fig. 2 in the variables $x=b^2/N$, $y=\beta/(1-\beta)$. Here, additional numerical data are given for $x>8$. Together with a satisfactory scaling behavior for the whole range of $x$ there is a remarkable linear dependence up to $x\approx 10$. The dashed line corresponds to $y=\gamma x$, with $\gamma \approx 1.4$.

In the large $N$-limit the distribution becomes Gaussian, a signature of the random nature of the eigenvectors:

$$w(u_k) \rightarrow \left[\frac{N}{2\pi}\right]^{1/2} \exp\left[-\frac{N}{2}u_k^2\right].$$

The entropy (3) of a GOE eigenvector, taking into account that all components are equally distributed, has the average value

$$H_{\mathrm{GOE}} = -\int_{-1}^{1} w(u)u^2 \ln u^2 du = \psi(\tfrac{1}{2}N+1) - \psi(\tfrac{3}{2}), \tag{5}$$

where $\psi(z)$ is the digamma function.

We then define the "entropy localization length" of eigenvectors of BRM as

$$l_H = N\exp(H - H_{\mathrm{GOE}}). \tag{6}$$

This definition of localization length is in close agreement with the more intuitive notion of localization as the shortest sequence of components carrying most of the vector's normalization.[18] The entropy localization length (6) has large fluctuations from one eigenvector to the other in the same matrix, although it clearly shows an average dependence on the eigenvalues. Moreover, for large matrices, its average over all the eigenvectors is very stable [for the GOE it may be shown that the fluctuations of the entropy around the average value (5) are of the order $\ln N/N$]. This property justifies the numerical computation of ensemble averages over a limited number of matrices (in our computations, twenty for $N=200$, ten for $N=400$, three or more for $N>600$).

We now introduce the average localization length

$$d = N\exp(\langle H\rangle - H_{\mathrm{GOE}}), \tag{7}$$

where the averaging is performed over all eigenvectors of one matrix, and over a number of different matrices of the ensemble.

The main quantity that undergoes scaling is $\beta = d/N$, the "scaled localization length," which takes values between 1 for the GOE and 0 for diagonal matrices in the large-$N$ limit.

The main results of our numerical computations are shown in Figs. 2-4. In Fig. 2 we plot the numerically obtained values of $\beta$ as a function of $x=b^2/N$, for $N$ ranging between 200 and 1000. It is seen that, with a good accuracy, all points fall on a smooth curve. Even tridiagonal matrices roughly follow the scaling curve (see Fig. 3).

It is illuminating to plot our numerical data in the new variables $\ln x$ and $\ln y$, with

$$y = \beta/(1-\beta). \tag{8}$$

In these new variables there is a remarkable linear behavior, $\ln(y) = \alpha \ln(x) + c$, that holds up to $x \approx 10$. A fit of the numerical data gives $\alpha \approx 1$ and $c \approx 0.35$. As an analytical expression for the scaling curve in the above region, we may therefore take

$$\beta = \gamma x/(1+\gamma x), \quad \gamma \approx 1.4. \tag{9}$$

This curve is plotted also in Fig. 2 and gives an excellent description of the numerical data. It is important to remark that for $x \ll 1$ we get from (9), $\beta \approx 1.4x$, which ex-

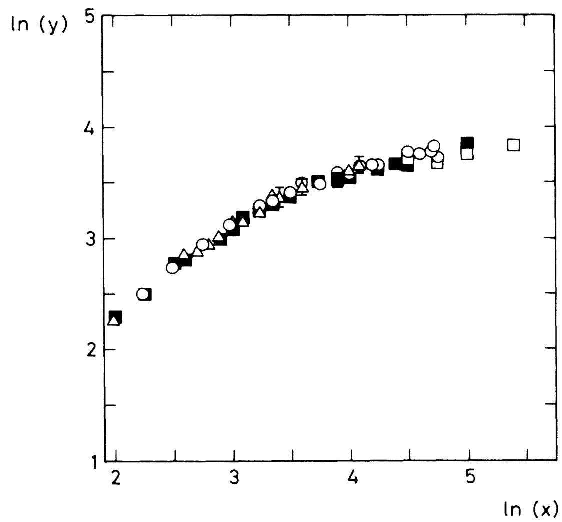


FIG. 4. Magnification of Fig. 3 for $\ln x \geq 2$. A few typical error bars are also shown.

actly follows from the theory of the kicked rotator once one identifies $x$ with $k^2/N$.[1]

The numerical exploration of the scaling curve at values $x \gg 1$ becomes increasingly difficult due to small denominators in (8), and requires larger and larger matrices. Moreover, when $b$ approaches $N/2$ the band structure is lost and the computed localization lengths deviate from the scaling line. However, it is seen (Figs. 3 and 4) that for $x \gg 1$ $(x > 10)$ numerical data deviate from the straight line, but nevertheless clearly indicate that the scaling behavior continues to hold. We presently do not know the analytical form of this asymptotic dependence. We also do not see any clear connection of the observed behavior with similar problems already discussed in solid-state physics.[22-24]

In this paper we have provided a convincing numerical evidence for the existence of a scaling behavior of the localization of eigenvectors of band random matrices. The scaling variable is $b^2/N$, where $b$ measures the size of the band. We want to stress that this scaling property is highly nontrivial and yet there is no explanation in the frame of matrix theory. We were led to conjecture this property not on the basis of mathematical considerations, but from the analogy with a physical model, the quantum kicked rotator. In addition, according to our preliminary computations,[25] a scaling behavior with the same scaling parameter holds also for the statistical properties of spectra, namely, the level- spacing distribution $P(s)$.

As a conclusive remark, we would like to note that the two distinct regimes in the scaling behavior of $\beta$ as a function of $x$ (Fig. 3) may have a counterpart in the so-called insulator and conductor regimes of disordered models. It is intriguing that, formally, the variable (8) is analogous to the electrical resistance, with $\beta$ playing the role of the transmission coefficient.

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