New Phase of Matter in Lamellar Phases of Tethered, Crystalline Membranes

John Toner

IBM Research Division, T. J. Watson Research Center, P.O. Box 218, Yorktown Heights, New York 10598

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A qualitatively new, "decoupled" phase of matter occurs in lamellar phases of tethered, crystalline membranes at large membrane separations d. This decoupled phase has no first-order elastic couplings between successive membranes; as a result, translational order falls off algebraically, with a universal exponent transverse to the membranes. The universal d dependence of the elastic constants is calculated, as is the critical separation d_c at which the decoupling phase transition occurs. *Untethered,* crystalline membranes melt, as d is increased, at some $d < d_c$.

Thin crystalline membranes have very unusual elastic properties.[1-3] Thermally excited transverse fluctuations of a membrane renormalize its in-plane Lamé coefficients μ and λ to zero at long wavelengths,[2] while its bend rigidity κ is driven to infinity.[2,3]

These theoretical ideas may be tested in lamellar phases of surfactant bilayers in multicomponent fluids,[4-6] which consist of a periodic stack of thin lamellae of surfactant molecules (of thickness $\delta \sim 30$ Å) separated by larger regions of liquid.

This paper treats lamellar phases of *tethered, crystalline* membranes. In practice, tethering can be accomplished by polymerizing the membranes.[7] My principal result is that, as the mean spacing d between successive membranes is increased, the system of membranes undergoes an equilibrium phase transition, from a conventional three-dimensional solid for $d < d_c$ to a "decoupled" phase for $d > d_c$. Typically, $d_c \sim 1000$ Å, which is experimentally accessible.[4-6]

The two phases differ in their elastic energies. The 3D solid phase is described by the usual elastic theory for a uniaxial solid.[8] In the decoupled phase, the elastic constant $c_{55} = 0$, *exactly*, and each membrane acts like an independent 2D solid. When c_{55} vanishes, the bending energy[9] $(K/2) \int d^3r (\nabla_\perp^2 u_z)^2$, where u_z is the displacement field orthogonal to the membranes and K is the bulk layer bending stiffness, is important.

In both the 3D solid and the decoupled phase, all of the elastic constants, except c_{55}, decrease algebraically at large d. Specifically, $c_{66} \propto c_{11} \propto d^{\tau-5}$, $c_{13} \propto d^{-5}$, $c_{33} \propto d^{-\tau-1}$, and $K \propto d^{\tau-3}$. Here $\tau = 3.1 \pm 0.2$ is the universal exponent for the entropic interaction between membranes.[10] At large d in the 3D solid phase, $c_{55} \propto d^{-\gamma_{55}}$, with

$$\gamma_{55}(d) = \frac{2\tau}{2 - \eta_{2D}^{ren}(G_0, d)} - 1, \qquad (1)$$

where the nonuniversal exponent $\eta_{2D}^{ren}(G_0, d) = 2(d/d_c)^{4-\tau}$ for d far from d_c. Near d_c, $\eta_{2D}^{ren}(G_0, d) = 2 - C \times (1 - d/d_c)^{1/2}$ as $d \to d_c^-$, with C a small constant of order $(\delta/d_c)^{\tau/2} \ll 1$, where δ is the membrane thickness.

This implies that as $d \to d_c^-$,

$$c_{55} \propto \exp\left(-\frac{O(1)}{C(1 - d/d_c)^{1/2}}\right).$$

I emphasize that c_{55} can get so small that the 3D solid phase may be experimentally indistinguishable from the decoupled phase. This will be described elsewhere.[11]

The elastic constants can be determined either through direct mechanical measurements, or correlation functions. The 3D solid is characterized by perfectly sharp Bragg peaks, since the Debye-Waller factor with $c_{55} \neq 0$ is finite. In the decoupled phase, the Debye-Waller factor diverges, and translational order decays algebraically.

Order transverse to the membranes is measured directly by x-ray scattering $S(\mathbf{q})$ at wave vectors $\mathbf{q}$ near $mG_\parallel \hat{z}$, where $G_\parallel \equiv 2\pi/d$. As in a smectic-$A$ liquid crystal,[12,13] in the decoupled phase $S(\mathbf{q})$ has the form $S(\mathbf{q}) \propto |\mathbf{q} - mG_\parallel \hat{z}|^{-2+\eta_{SM}(m)}$ for $\mathbf{q} \parallel \hat{z}$ and $S(\mathbf{q}) \propto |\mathbf{q} - mG_\parallel \times \hat{z}|^{-4+2\eta_{SM}(m)}$ for $\mathbf{q} - mG_\parallel \hat{z} \perp \hat{z}$. Here $\eta_{SM}(m) = \eta_0 m^2$, and $\eta_0 = k_B T G_\parallel^2 / 8\pi (BK)^{1/2}$, where

$$B(d) = c_{33}(d) - \frac{c_{13}^2(d)}{2c_{66}(d) + c_{11}(d)}.$$

As in fluid membranes,[14] η_0 is universal.

The translational order within each membrane can be described by expanding the two-dimensional density ρ_2 within the membrane in a Fourier series:

$$\rho_2(\mathbf{r}_\perp) = \rho_2^0 + \sum_{\{G\}} \rho_G(\mathbf{r}_\perp) e^{i\mathbf{G}_\perp \cdot \mathbf{r}_\perp},$$

where $\{G_\perp\}$ is the set of reciprocal-lattice vectors of the membrane. I find that the correlation function $C_G(\mathbf{r}) \equiv \langle \rho_G^*(\mathbf{r}) \rho_G(0) \rangle$ has three regimes of behavior:

$$C_G(\mathbf{r}) \propto (r/a)^{-\eta_{2D}^0(G)}, \qquad r \ll \xi_\mu(T), \qquad (2)$$

$$C_G(\mathbf{r}) \propto \exp[-a\eta_{2D}^0(r/\xi_\mu)^{\eta_\perp}], \qquad \xi_\mu \ll r \ll L_\perp(d), \qquad (3)$$

$$C_G(\mathbf{r}) \propto (r/a)^{-\eta_{2D}^{ren}(G,d)}, \qquad r \gg L_\perp(d), \qquad (4)$$

where μ_0 and λ_0 are the bare areal Lamé coefficients of a single membrane, the exponent $\eta_{2D}^{ren}(G,d)$ is always > 2

in the decoupled phase,

$$\eta_{2D}^0(\mathbf{G}) = \frac{k_B T |\mathbf{G}|^2}{4\pi\mu_0} \frac{3\mu_0 + \lambda_0}{2\mu_0 + \lambda_0},$$

$$\eta_{2D}^{ren}(\mathbf{G},d) = \frac{k_B T |\mathbf{G}|^2}{4\pi c_{66}(d)d} \frac{3c_{66}(d) + c_{11}(d)}{2c_{66}(d) + c_{11}(d)} \propto d^{4-\tau},$$

$$\eta_\perp = \frac{8}{\tau} - 2 = 0.58 \pm 0.17, \quad \xi_\mu = \frac{\kappa_0}{(\mu_0 k_B T)^{1/2}},$$

$$L_\perp(d) = \xi_\mu \left(\frac{d^2\mu_0}{\kappa_0}\right)^{\tau/4},$$

and $\alpha = O(1)$.

The form of the in-plane scattering $S(\mathbf{q}_\perp)$ near $\mathbf{q}_\perp = \mathbf{G}$ implied by (2)–(4) is

$$S(\mathbf{q}_\perp) \propto |\mathbf{q}_\perp - \mathbf{G}|^{-2+\eta_{2D}^0(\mathbf{G})}, \quad |\mathbf{q}_\perp - \mathbf{G}| \gg \xi_\mu^{-1}, \quad (5)$$

$$S(\mathbf{q}_\perp) \propto |\mathbf{q}_\perp - \mathbf{G}|^{-2-\eta_\perp}, \quad \xi_\mu^{-1} \gg |\mathbf{q}_\perp - \mathbf{G}| \gg \Delta_\mathbf{G}, \quad (6)$$

$$S(\mathbf{q}_\perp) \propto \text{const} + |\mathbf{q}_\perp - \mathbf{G}|^{-2+\eta_{2D}^{ren}(\mathbf{G},d)}, \quad (7)$$

$$|\mathbf{q}_\perp - \mathbf{G}| \ll L_\perp^{-1}(d),$$

where $\Delta_\mathbf{G} = [\eta_{2D}^0(\mathbf{G})]^{1/\eta_\perp} \xi_\mu^{-1} \propto |\mathbf{G}|^{2/\eta_\perp}$. Note the surprising minus sign on $\eta_\perp$ in Eq. (6). Note also that, since $\eta_{2D}^{ren}(\mathbf{G},d)$ is always > 2 in the decoupled phase, $S(\mathbf{q}_\perp)$ remains finite as $\mathbf{q}_\perp \to \mathbf{G}$; it may or may not have

a cusp, depending upon whether or not $\eta_{2D}^{ren}(\mathbf{G},d) < 3$. This structure is extended in rods in the q_z direction.

I will now sketch the derivation of these results, beginning by reviewing the unusual elastic properties[2,3] of an isolated crystalline, tethered membrane.

In such a membrane, the two-dimensional areal Lamé elastic constants μ and λ become wave-number-q-dependent quantities,[2] both vanishing as $q \to 0$ like $q^{\eta_\perp}$, where $\eta_\perp = 0.58 \pm 0.17$ is a universal exponent.[2,10] Likewise, the bend modulus κ that determines the energy cost of bending the membrane $E_{bend} = \frac{1}{2}\int d^2r\, \kappa/R^2(r)$, where $R(r)$ is the local radius of curvature, diverges[2,3] as $q \to 0$ like[2] $q^{-\eta}$, with[11] $\eta \equiv \frac{1}{2}(2-\eta_\perp) = 0.71 \pm 0.08$.

The mean-squared real-space thermal fluctuation $\langle h^2(\mathbf{r})\rangle|_L$ of the transverse displacement h of a membrane of lateral extent L is

$$\langle h^2(\mathbf{r})\rangle|_L = k_B T \int_{q > L^{-1}} \frac{d^2q}{\kappa(q)q^4} = O(1) \times \frac{k_B T}{\kappa_0} L^{2-\eta}\xi_\mu^\eta,$$

where I have used the fact[2] that $\kappa(q) = \kappa_0 (q\xi_\mu)^{-\eta} \times O(1)$ for $q\xi_\mu \ll 1$, and κ_0 is the "bare" (short-distance) value of κ.

The unusual form of the in-plane correlation function $C_\mathbf{G}(\mathbf{r})$ given in Eq. (3) is due to the vanishing of μ and λ. Using the phase relation $\rho_\mathbf{G}(\mathbf{r}) = \rho_\mathbf{G}^0 e^{i\mathbf{G}\cdot\mathbf{u}(\mathbf{r})}$, where $\rho_\mathbf{G}^0$ is a constant, leads to $C_\mathbf{G}(\mathbf{r}) \propto \langle e^{i\mathbf{G}\cdot[\mathbf{u}(\mathbf{r})-\mathbf{u}(0)]}\rangle$. In the harmonic approximation,

$$C_\mathbf{G}(\mathbf{r}) \propto \exp\{-\tfrac{1}{2}\langle |\mathbf{G}\cdot[\mathbf{u}(\mathbf{r})-\mathbf{u}(0)]|^2\rangle\} = \exp\left[-\frac{|\mathbf{G}|^2 k_B T}{2}\int \frac{d^2q}{(2\pi)^2}\frac{1-\cos(\mathbf{q}\cdot\mathbf{r})}{\mu q^2}\left(\frac{3\mu+\lambda}{2\mu+\lambda}\right)\right]. \quad (8)$$

In this harmonic approximation, the integral in (8) is logarithmic in r, and[15] $C_\mathbf{G}(\mathbf{r}) \propto r^{-\eta(\mathbf{G})}$ with $\eta(\mathbf{G}) = (|\mathbf{G}|^2 k_B T/4\pi\mu)(3\mu+\lambda)/(2\mu+\lambda)$. Since anharmonic effects are unimportant[2,3] for $r \ll \xi_\mu$, this power-law result is valid on those length scales, with μ and λ given by their bare values μ_0 and λ_0, respectively. This implies Eq. (2). For $r \gg \xi_\mu$, the anomalous elasticity invalidates this power-law result. However, Eq. (8) still remains valid, *provided* that one replaces the *constants* μ and λ with the renormalized, wave-vector-dependent quantities[2] $\mu(q) = \mu_0(q\xi_\mu)^{\eta_\perp}\times O(1) = \lambda(q)\times O(1)$ for $q\xi_\mu \ll 1$. Using these in Eq. (8) gives Eq. (3) for $C_\mathbf{G}(\mathbf{r})$.

Once a membrane is one member of a stack of identical membranes of mean spacing d, its perpendicular fluctuations $\langle h^2\rangle^{1/2}$ are bounded, since it will bump into its neighbors once $\langle h^2\rangle^{1/2} > d$. This will happen if the lateral extent L of the membranes exceeds a critical value $L_\perp(d)$ at which $\langle h^2\rangle^{1/2} = d$. Using

$$\langle h^2(\mathbf{r})\rangle|_L = (k_B T/\kappa_0)L^{2-\eta}\xi_\mu^\eta \times O(1),$$

this implies $L_\perp(d) = \xi_\mu(d^2\mu_0/\kappa_0)^{1/(2-\eta)}$.

$L_\perp(d)$ is also roughly the distance between points of contact of two neighboring fluctuating membranes.[14] Assigning to each contact a free energy of order $k_B T$ (since $k_B T$ sets the free-energy scale for entropic effects), one obtains an entropic interaction[10,14] per unit area between

neighboring membranes given by

$$V_{ent}(d) = \frac{k_B T}{L_\perp^2(d)} \times O(1)$$

$$= O(1) \times \frac{k_B T}{\xi_\mu^2}\left[\frac{\kappa_0}{\mu_0 d^2}\right]^{2/(2-\eta)} \propto d^{-\tau},$$

where[10] the exponent $\tau \equiv 4/(2-\eta)$. Combining this with the numerical simulation[10] result $\tau = 3.1 \pm 0.2$ gives $\eta = 0.71 \pm 0.08$, from which the scaling relation[2] $\eta_\perp = 2(1-\eta)$ gives $\eta_\perp = 0.58 \pm 0.17$.

The fact that the fractional change in the layer spacing $\delta d/d = \partial_z u_z$ implies[10]

$$c_{33}(d) = \frac{\partial^2 V_{ent}}{\partial d^2}d$$

$$= \frac{(k_B T)^2}{\kappa_0}\left[\frac{\kappa_0}{\mu_0 d^2}\right]^{(\tau+1)/2}\left[\frac{\mu_0}{\kappa_0}\right]^{3/2}$$

$$\times O(1) \propto d^{-\tau-1}.$$

The suppression of transverse fluctuations of the membrane by its neighbors clearly affects the renormalization of the elastic constants μ, λ, and κ. In fact, the fluctuation contribution to these quantities coming from all

wave numbers $q > L_\perp^{-1}(d)$ now becomes finite,[16] since the transverse fluctuations are suppressed far below their free-membrane values at such wave numbers. Hence, the wave-number dependences of $\mu(q)$, $\lambda(q)$, and $\kappa(q)$ are cut off once $qL_\perp(d) \sim 1$; and all of these quantities become constants for $qL_\perp(d) \ll 1$. These constants do depend strongly on d, however. This dependence can be determined by matching the constant values $\mu(d)$, $\lambda(d)$, and $\kappa(d)$ for $qL_\perp \ll 1$ onto their wave-vector–dependent values for $qL_\perp \gg 1$ at $qL_\perp = 1$. I thereby obtain

$$\mu(d) = \mu_0 \left[\frac{L_\perp(d)}{\xi_\mu} \right]^{-\eta_\perp} \times O(1)$$

$$= \mu_0 \left[\frac{\mu_0 d^2}{\kappa_0} \right]^{(\tau-4)/2} \times O(1) \propto d^{\tau-4} \propto \lambda(d),$$

$$\kappa(d) = \kappa_0 \left[\frac{L_\perp(d)}{\xi_\mu} \right]^{\eta} \times O(1)$$

$$= \kappa_0 \left[\frac{d^2 \mu_0}{\kappa_0} \right]^{\tau/2-1} \times O(1) \propto d^{\tau-2}.$$

Because μ and λ are now constants at sufficiently large distances, the in-plane translational correlation function $C_G(r)$ will again become algebraic. This implies Eq. (4), with

$$\eta_{2D}^{\text{ren}}(G,d) = \frac{k_B T |G|^2}{4\pi\mu(d)} \frac{3\mu(d)+\lambda(d)}{2\mu(d)+\lambda(d)}$$

$$= 2 \left[\frac{d}{d_c} \right]^{4-\tau} \frac{|G|^2}{|G_0|^2}.$$

Here I have defined

$$d_c \equiv \lim_{d \to \infty} \left[\frac{8\pi\mu(d)d^{4-\tau}[2\mu(d)+\lambda(d)]}{k_B T |G_0|^2[3\mu(d)+\lambda(d)]} \right]^{1/(4-\tau)}$$

$$= \left[\frac{\kappa_0}{\mu_0} \right]^{1/2} \left[\frac{\mu_0 a^2}{k_B T} \right]^{1/(4-\tau)} \times O(1),$$

G_0 is the shortest reciprocal-lattice vector of the membrane, and $a \equiv 2\pi/|G_0|$.

The divergence of $\eta_{2D}^{\text{ren}}(G,d)$ implies the existence of the decoupled phase for $d > d_c$. To see this, first note that the most general coupling between the translational order parameters $\rho_G^{(1)}(r)$ and $\rho_G^{(2)}(r)$ in successive membranes is[17]

$$H_c = -\sum_G J_G(d) \int d^2r \{[\rho_G^{(1)}(r)]^* \rho_G^{(2)}(r) + \text{c.c.}\}. \quad (9)$$

Quite generally, the renormalization-group eigenvalue γ_g of the coupling strength of any operator $g \int d^d r \, O(r)$ added as a perturbation to a Hamiltonian is given by $\gamma_g = d - \eta_O/2$, where η_O is defined via $\langle O(r)O(0)\rangle|_{g=0} \propto r^{-\eta_O}$. For J_G, $O(r) = [\rho_G^{(1)}(r)]^* \rho_G^{(2)}(r) + \text{c.c.}$ Thus

the relevant correlation function is

$$\langle [\rho_G^{(1)}(r)]^* \rho_G^{(2)}(r)\rho_G^{(1)}(0)[\rho_G^{(2)}(0)]^* \rangle |_{J_G=0}$$

$$= \langle [\rho_G^{(1)}(r)]^* \rho_G^{(1)}(0)\rangle^2 |_{J_G=0}$$

$$= C_G^{(2)}(r) \propto r^{-2\eta_{2D}^{\text{ren}}(G,d)},$$

where I have used the fact that in the unperturbed ($J_G=0$) Hamiltonian $\rho_G^{(1)}$ and $\rho_G^{(2)}$ are decoupled. Thus, in this case $\eta_O = 2\eta_G^{\text{ren}}$, whence the renormalization-group eigenvalue γ_G of J_G is $\gamma_G = 2 - \eta_{2D}^{\text{ren}}(G,d)$. Since $\eta_{2D}^{\text{ren}}(G,d)$ diverges as $d \to \infty$ for any temperature T, and for any material parameters μ_0, λ_0, and κ_0, there always exists a critical distance d_c beyond which even the largest γ_G is less than zero;[18] namely, that at which $\eta_{2D}^{\text{ren}}(G_0,d_c) = 2$. This criterion is readily seen to lead precisely to the d_c defined with such foresight earlier.

For $d > d_c$, all of the γ_G's are < 0, all of the J_G's renormalize to zero, and in-plane translational order in successive membranes is effectively decoupled. For $d < d_c$, successive membranes are coupled, and the system is a conventional three-dimensional solid. Thus there is an equilibrium phase transition at $d = d_c$ between the two.

For untethered (i.e., unpolymerized) membranes, an in-plane melting transition necessarily pre-empts this decoupling transition, because isolated 2D crystals with $\eta(G_0) > \frac{1}{3}$ melted[15] by dislocations [recall $\eta_{2D}^{\text{ren}}(G_0,d) > 2$ in the decoupled phase].

This melting would be interesting in its own right, and would show up as a 3D-solid to stacked-hexatic[19] transition. However, dislocations can be excluded from the membranes by polymerizing[7] them, thereby forbidding bond breaking.

The elastic constants other than c_{33} can be computed as follows.

The in-plane elastic constants c_{66} and c_{11} are just the bulk analogs of the two-dimensional elastic constants μ and λ computed earlier, and are given by

$$c_{66}(d) = \frac{\mu(d)}{d} = \frac{\mu_0}{d} \left[\frac{d^2\mu_0}{\kappa_0} \right]^{\tau/2-2} \times O(1) \propto d^{\tau-5},$$

and $c_{11}(d) = \lambda(d)/d = c_{66}(d) \times O(1)$. Likewise, the bulk bend elastic constant

$$K(d) = \frac{\kappa(d)}{d} = \frac{\kappa_0}{d} \left[\frac{d^2\mu_0}{\kappa_0} \right]^{\tau/2-1} \propto d^{\tau-3}.$$

c_{13} can be calculated by first noting that contacts between two neighboring membranes must locally change the in-plane lattice constants a near the point of contact by some amount Δa_c (since no symmetry forbids such a change). Thus the average in-plane layer constant $a(d)$ should obey $a(d) = a(\infty) + \Delta a_c A_{\text{contact}}/L_\perp^2(d)$, where A_{contact} is the area around a point of contact over which the lattice constant is affected. A simple argument shows that $A_{\text{contact}} = \kappa_0 r_l^2/k_B T$, where r_l is the range of

the interatomic forces. Thus I obtain $a(d) = a(\infty) + \Delta a_c$ $\times \kappa_0 r_f^2 / k_B T L_\perp^2(d)$. Now consider compressing the membranes closer together, which requires making $\partial_z u_z \neq 0$. Using the fact that $\partial_z u_z = \partial d/d$, one finds that the change $\Delta a(\partial_z u_z)$ in the average lattice constant is

$$\Delta a = \frac{\partial a}{\partial d} d\partial_z u_z = -\tau \Delta a_c \frac{\kappa_0 r_f^2}{k_B T L_\perp^2(d)} \partial_z u_z .$$

Since $\mathbf{V}_\perp \cdot \mathbf{u}_\perp = \Delta a/a$, I find that $\mathbf{V}_\perp \cdot \mathbf{u}_\perp = -[\tau \Delta a_c \kappa_0 r_f^2 / a k_B T L_\perp^2(d)]\partial_z u_z$. On the other hand, minimizing the elastic energy over $\mathbf{u}_\perp$ for fixed $\partial_z u_z \neq 0$ gives $\mathbf{V}_\perp \cdot \mathbf{u}_\perp = -(c_{13}/c_{11})\partial_z u_z$. Equating these two expressions gives

$$c_{13} = c_{11} \frac{\kappa_0 r_f^2}{k_B T L_\perp^2(d)} \times O(1) = \frac{\kappa_0 r_f^2}{d^5} \times O(1) ,$$

assuming $\Delta a_c = O(a(\infty))$.

Finally, c_{55} can be expressed in terms of the inter-membrane couplings $J_G(d)$ in Eq. (9). Writing $\rho_G^{(1,2)} = \rho_G^0 \exp(i\mathbf{G} \cdot \mathbf{u}_\perp^{(1,2)})$, that equation can be rewritten in terms of the in-plane displacements $\mathbf{u}_\perp$:

$$H_c = \sum_G J_G |\rho_G|^2 \int \cos[\mathbf{G} \cdot (\mathbf{u}_\perp^{(1)} - \mathbf{u}_\perp^{(2)})]d^2 r .$$

Expanding the cosine for small $\mathbf{G} \cdot (\mathbf{u}_\perp^{(1)} - \mathbf{u}_\perp^{(2)})$ and comparing the result with conventional uniaxial elastic theory gives $c_{55} = \sum_G J_G(d) |\rho_G|^2 |\mathbf{G}|^2 d$. For large d, $J_G(d)$, like the entropic interaction V_{ent} described earlier, is clearly dominated by the points of contact between adjacent membranes. This implies that $J_G(d) = E_G/L_\perp^2(d)$, where E_G is a d-independent energy scale.

This "bare" J_G will be further renormalized, with renormalization-group eigenvalue γ_G, by in-plane fluctuation effects. As a result, the physical, renormalized $J_{G_0}^{phys}$ is given by

$$J_{G_0}^{phys} = \left[\frac{E_{G_0}}{(L_\perp |G_0|)^2 k_B T} \right]^{2/\gamma_{G_0}} k_B T |G_0|^2 ,$$

and hence $c_{55}(d) \propto d^{-\gamma_{55}(d)}$ with

$$\gamma_{55}(d) = \frac{2\tau}{\gamma_{G_0}} - 1 = \frac{2\tau}{2 - \eta_{2D}^{ren}(G_0,d)} - 1 .$$

A renormalization-group analysis to higher order in J_{G_0} leads[11] to the results for $\eta(G_0,d)$ and $\gamma_{55}(d)$ near the transition summarized earlier.

Integrating the variable $\mathbf{u}_\perp$ out of the partition function $Z \equiv \int D\mathbf{u}_\perp Du_z e^{-\beta H(\mathbf{u}_\perp, u_z)}$ when $c_{55} = 0$ leaves an effective elastic Hamiltonian for u_z alone of the canonical smectic form

$$H_{eff}(u_z) = \frac{1}{2} \int d^3 r [B(d)(\partial_z u_z)^2 + K(d)(\nabla_\perp^2 u_z)^2] ,$$

with

$$B(d) = c_{33}(d) - \frac{c_{13}^2(d)}{2c_{66}(d) + c_{11}(d)} \propto d^{-\tau - 1} .$$

This smectic form for the elastic energy for u_z immediately implies that all static correlation functions involving u_z in the decoupled phase are the same as in a smectic phase. Using the expressions for the various elastic constants derived here, one can show that the smectic exponent η_0 is universal as $d \to \infty$, as asserted earlier.

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