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Localization Decay Induced by Strong Nonlinearity in Disordered Systems

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The scattering of a nonlinear wave packet as an envelope soliton by a one-dimensional disordered system is studied. It is well known that in the linear limit the transmission coefficient decays exponentially with a characteristic localization length. We predict, using a simple independent scattering approach and soliton perturbation theory in the framework of the nonlinear Schrödinger equation, that strong nonlinearity above a certain threshold allows undistorted propagation of wave packets.

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Wave propagation in nonlinear disordered media has become an extensively studied subject in recent years[1-4] because of the complex properties and qualitatively new effects arising from the competition of disorder and nonlinearity. Disorder in a linear chain generally originates an exponential decay of the transmission coefficient (Anderson localization, see, e.g., Ref. 5) that can be easily extended to phonons, acoustic, and electromagnetic waves, etc. It is also well known that, in the absence of disorder, some nonlinear systems show excitations in the form of localized wave packets (solitons) that propagate without changes in their shape or velocity. It has been shown that weak nonlinearity acting against disorder changes the exponential-length dependence of the transmission coefficient into a power-law one.[1,2,4] This Letter aims to demonstrate that if the system's nonlinearity is strong, propagation of nonlinear wave packets as solitary waves in disordered systems may be undistorted, i.e., localization effects completely decay because of nonlinearity and, moreover, that there exists a threshold value for this possibility of propagation.

We start from the dimensionless nonlinear Schrödinger (NLS) equation for the wave variable $u(x,t)$,

$$iu_t + u_{xx} + 2|u|^2 u = \epsilon(x) u \,, \tag{1}$$

that arises in this form in a number of problems in solid-state physics (see, e.g., Refs. 6–8); as an example, it can be derived in the small-amplitude limit for the sine-Gordon and ϕ^4 models.[9] The term on the right-hand side of Eq. (1), $\epsilon(x) \equiv \epsilon \sum_n \delta(x - x_n)$, describes random-point impurities with equal intensities ϵ and random positions x_n, and it may represent, e.g., the structural disorder of the associated system.

If we restrict ourselves to the linear problem neglecting the last term on the left-hand side of the Eq. (1), the study of the propagation of monochromatic waves in a randomly inhomogeneous medium leads to the stochastic differential equation $-u_{xx} + \epsilon(x) u = k^2 u$, where k is the wave number, and to the important phenomenon of localization of states by random inhomogeneities due to scattering.[5] Localization means that the transmission coefficient T decays exponentially with the system length L. If $\epsilon(x)$ is a stationary ergodic random process, then a positive finite number exists, the so-called localization length $\lambda(k)$, such that $L^{-1} \ln T(k) \sim -\lambda^{-1}(k)$, and hence for large L [significantly wider than $\lambda(k)$] very little transmission is allowed. Thus, in the linear Schrödinger equation,[5,10] if the conditions $\epsilon^2 \ll k^2 \ll \epsilon^2 L p$ and $p \ll k$ hold, the localization length turns out to be $\lambda(k) = 4k^2/p\epsilon^2$; p^{-1} has the sense of a mean distance between impurities.

The linear equation also leads to a decay of the transmission coefficient of linear wave packets. The mean value τ may be defined for the whole packet as follows,

$$\tau = \int_0^\infty dk\, P(k - k_0) \langle T(k) \rangle \,, \tag{2}$$

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where $P(k)$ describes the spectral structure of the packet, k_0 is the carrier wave number, and

$$\langle T(k)\rangle = \frac{\pi^{5/2}}{2}\left[\frac{L}{\lambda(k)}\right]^{-3/2}\exp\left[\frac{-L}{4\lambda(k)}\right] \quad (3)$$

is the mean transmission coefficient with $\lambda(k)$ defined above. Using the results of Refs. 5 and 10 it can be demonstrated that in a general case the dependence of τ on the length system L is always exponential but the (positive) power of L in the exponent depends on the relation between the parameters of the wave packet. So, an exponential decay of the transmission coefficient in the linear system characterizes both a single linear wave and a linear wave packet.

Let us now consider the nonlinear case. The homogeneous nonlinear system (1) allows the distortionless propagation of localized excitations in the form of envelope solitons,

$$u_s(x,t) = a\frac{\exp\{-(iV/2)x - i[(V^2/4)-a^2]t\}}{\cosh[a(x-Vt)]}, \quad (4)$$

where a is the soliton amplitude and V is the speed. We consider the scattering of the soliton (4) by a random system of point impurities with equal intensities ϵ. A soliton incides on the disordered layer from the left, and it decomposes into reflected (r) and transmitted (t) parts. After passing through each impurity the wave packet will reorganize itself and become again a soliton plus some small-amplitude waves. We can use the change of the two NLS integrals of motion, the energy E and the "number of quasiparticles" N, defined by

$$E = \int_{-\infty}^{\infty} dx[|u_x|^2 + \epsilon(x)|u|^2 - |u|^4], \quad N = \int_{-\infty}^{\infty} dx|u|^2, \quad (5)$$

to describe the process through two magnitudes: the total-energy transmission coefficient $T^{(E)} = E_t/E_i$, that is, the transmitted energy E_t over the incident one E_i, and the "number-of-particles" transmission coefficient $T^{(N)} = N_t/N_i$. Of course, the constraints $E_i = E_t + E_r$ = const and $N_i = N_t + N_r$ = const hold.

When the concentration p of impurities is low, the average distance between two nearby impurities is larger than the soliton size. In this limit we may treat the scattering by many impurities independently, $T \approx \prod_j T_j$, T_j being the transmission coefficient of the jth impurity. It should be noticed that even in this approach randomness is still present through the use of mean transmission coefficients.

The transmitted soliton for the jth impurity is then the incident one for the $(j+1)$th scatterer. So, we can write (cf. Ref. 3)

$$E_{j+1} = E_j T_j^{(E)}(E_j,N_j), \quad N_{j+1} = N_j T_j^{(N)}(E_j,N_j), \quad (6)$$

and

$$\Delta E_{j+1} = E_{j+1} - E_j = -E_j R_j^{(E)}(E_j,N_j), \quad (7)$$

$$\Delta N_{j+1} = N_{j+1} - N_j = -N_j R_j^{(N)}(E_j,N_j), \quad (8)$$

where $R_j^{(E,N)} = 1 - T_j^{(E,N)}$ stand for energy and number-of-particles reflection coefficients. These coefficients can be calculated, for $\epsilon \ll 1$, by employing the soliton perturbation theory based on the inverse scattering transform.[8] The value $R^{(N)}$ was already presented in detailed form in Ref. 11. The expressions for these coefficients are

$$R^{(N)} = \frac{\pi\epsilon^2}{64NV}\int_0^\infty dy\, F(y,a), \quad (9)$$

$$R^{(E)} = \frac{\pi\epsilon^2 V}{256E}\int_0^\infty dy\, y^2 F(y,a), \quad (10)$$

where

$$F(y,a) = \frac{[(y+1)^2+a^2]^2}{\cosh^2[(\pi/4a)(y^2+a^2-1)]}, \quad (11)$$

and $a \equiv N/V$. These results obtained in the Born approximation are valid if $\epsilon \ll 1$ and $V^2 \gg |\epsilon| a$ (see Ref. 11).

Using (2) and (4) and taking into account that there are $(\Delta x)p$ impurities in the interval Δx and that the soliton energy and number of particles are functions of a and V (see, e.g., Refs. 8 and 11) given by $N = 2a$ and $E = \frac{1}{4}N(V^2 - \frac{1}{3}N^2)$ we are able to derive from (7) and (8) the following equations:

$$\frac{dN}{dz} = -\frac{1}{V}\int_0^\infty dy\, F(y,a), \quad (12)$$

$$\frac{dV}{dz} = -\frac{1}{2N}\int_0^\infty dy(y^2-1)F(y,a) - \frac{N}{2V^2}\int_0^\infty dy\, F(y,a), \quad (13)$$

where the distance is measured in units of $x_0 = 64/\pi p\epsilon^2$, i.e., $z = x/x_0$.

In the linear limit, $a \ll 1$, the system (12) and (13) can be solved analytically yielding $V(x) = V(0)$ = const, and hence

$$T^{(N,E)}(x) = N(x)/N(0) = E(x)/E(0) = e^{-x/\lambda_0}, \quad (14)$$

where $\lambda_0 \equiv V^2(0)/p\epsilon^2 = 1/pR_1$, R_1 being the reflection coefficient of one impurity. This result demonstrates the exponential decay of the transmission coefficient. As we can see, these equations show again the same behavior as the one of the linear problem, where $\lambda_0 = \lambda(k_0)$, and k_0 has the sense of a carrier wave number of the packet.

In the case $a \gtrsim 1$ the system was studied numerically, employing the usual rectangles method to estimate the integrals and Euler's procedure to integrate the equations; some cases were verified with a leap-frog scheme and the result was always fully satisfactory. In this way, we concluded that the asymptotic change in $T^{(N,E)}(z)$ depends essentially on the value of the parameter $a(0)$

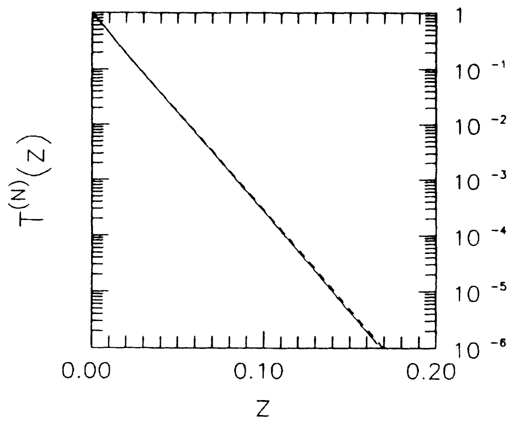

FIG. 1. The transmission coefficient $T^{(N)}(z)=N(z)/N(0)$ vs z when initial conditions are $N(0)=0.01$, $V(0)=0.5$ [$\alpha(0)=0.02$] (solid line is numerical; dashed line is analytical).

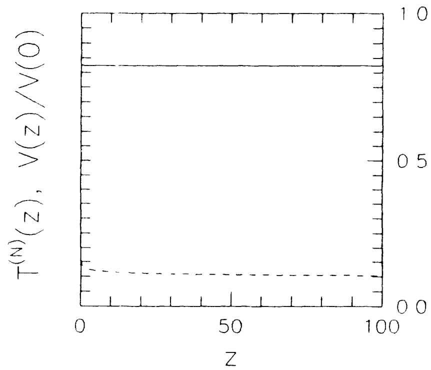

FIG. 3. Same as in Fig. 2 when $N(0)=1.25$, $V(0)=0.5$ [$\alpha(0)=2.5$]. The initial decreasing is too fast to be clearly appreciated at this scale.

$=N(0)/V(0)$ that is physically related to the nonlinearity of the incoming wave: The greater α is, the larger the number of quasiparticles in the soliton becomes, and the smaller its spatial extension; on the contrary, if α is small, the wave looks very similar to a linear wave packet. It can be simply proved by computing the derivative of $\alpha(z)$ that the solution α_c of the transcendental equation $\alpha_c^2-2+G(\alpha_c)=0$, with

$$G(\alpha)\equiv\int_0^\infty dy(y^2-1)F(y,\alpha)\Big/\int_0^\infty dy\,F(y,\alpha)\,,$$

is such that $\alpha(0)=\alpha_c$ implies $\alpha(z)=\alpha_c$ along the whole disordered layer. Even more, solutions of the system (12) and (13) verify that $\alpha(z)$ is monotonically, strictly increasing [decreasing] if $\alpha(0)>\alpha_c$ [$\alpha(0)<\alpha_c$]. We found by solving approximately the equation and also by integrating the system (12) and (13) that $\alpha_c\approx 1.28505$. Therefore, for initial conditions such that $\alpha(0)\ll\alpha_c$, the system evolves, in perfect agreement with the previous analytical result (14), to a final state in which N tends exponentially to zero while V goes to a constant positive value, and hence satisfying $\alpha(\infty)=0$ as required (see Fig. 1). If $1\sim\alpha<\alpha_c$, the decay consists of an initial, slow transient and after that we find a fast exponential behavior (see Fig. 2). Finally, initial conditions with $\alpha(0)>\alpha_c$ lead to a situation in which both N and V become practically constant (see Fig. 3) and so does α, $\alpha(\infty)$ having some limit value around $\alpha\approx 10$. The possible behaviors of α are summarized in Fig. 4. The precise form of the dependence of the transmission coefficients on z is determined not only by α but also by the values of $N(0)$ and $V(0)$: In particular, the smaller $N(0)$ and $V(0)$ are, the smaller the interval needed to reach the asymptotic regime is, and so the initial slope can be roughly 10^5 for $N(0),V(0)\approx 0.1$, for instance. Anyway, we must stress that the general property of these curves is the same: Both transmission coefficients tend to their asymptotic constant, nonzero values.

To sum up, we have considered the nonlinear wavepacket scattering by a disordered system in the frame-

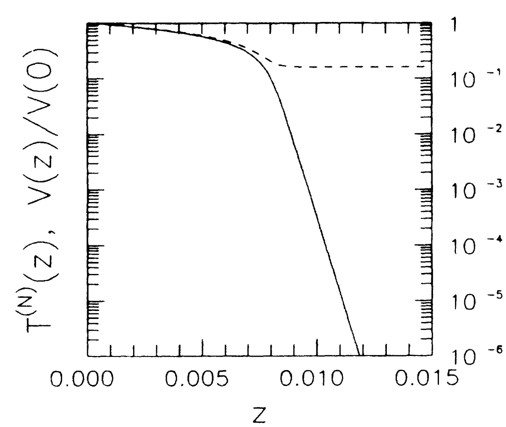

FIG. 2. The transmission coefficient $T^{(N)}$ (solid line) and the function $V(z)/V(0)$ (dashed line) when initial conditions are $N(0)=0.625$, $V(0)=0.5$ [$\alpha(0)=1.25$].

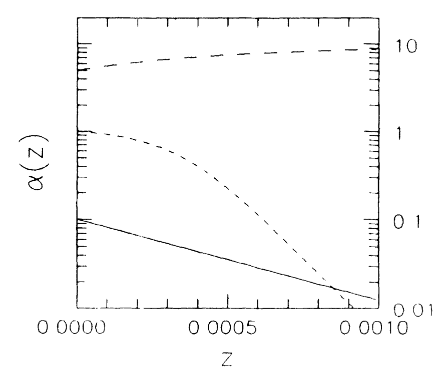

FIG. 4. The parameter $\alpha(z)$ for different choices of $\alpha(0)$. $V(0)=0.1$ in all three cases. Solid line, $\alpha(0)=0.1$ [$N(0)=0.01$]; short-dashed line, $\alpha(0)=1$ [$N(0)=0.1$]; long-dashed line, $\alpha(0)=5$ [$N(0)=0.5$].

work of the NLS equation, and demonstrated that strong nonlinearity can completely inhibit the localization effects stipulated by the disorder. It is a very remarkable feature that this effect appears over a threshold nonlinearity. Below this threshold, the transmission coefficient tends to zero as the size of the system increases, either exponentially (Fig. 1) or exponentially after a short transient (Figs. 1-4). Above the threshold value our model demonstrates undistorted motion of the nonlinear wave packet along the disorder system; i.e., the transmission coefficient does not decay (Fig. 3) and localization does not happen in the system anymore, probably because of the small soliton width for large values of α. It is rather striking the fact that we have not found power-law decays in any case; to this respect, we believe that if this is to happen in our model, it should do in the vicinity of α_c. It is also possible that this is a consequence of our simplifications, but we have not been able to get a definite conclusion about this point.

As a matter of fact, the effect can be quite more complicated. In our analytical considerations, we used a simple independent scattering approach and Born approximation of the perturbation theory. We believe that taking into account the additional small contributions, e.g., interference during the propagation, should provide a slow decreasing of the nonlinear transmission coefficient (probably, logarithmically) as the size of the system increases, so that the decay length may be regarded as the sum of a nonlinear value, λ_n, and a linear localization length, λ_l. λ_n depends mainly on the amplitude and it is actually very large (it characterizes the decreasing of the soliton amplitude to a small value which should be much less than unity). The soliton transforms into a linear wave packet due to scattering along a length of the order of λ_n, and then it scatters as a linear object, its transmission coefficient decaying exponentially. Since $\lambda_n \gg \lambda_l$, the real scattering of nonlinear wave packets for systems of length $L < \lambda_n$ must be very small.

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