Solution and Hidden Supersymmetry of a Dirac Oscillator

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We have found the complete energy spectrum and the corresponding eigenfunctions of the recently proposed Dirac oscillator. We found the electromagnetic potential associated with its interaction term. This exactly soluble problem has a hidden supersymmetry, responsible for the special properties of its energy spectrum. We calculate the related superpotential and we discuss the implications of this supersymmetry on the stability of the Dirac sea.

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In a recent work, Moshinsky and Szczepaniak[1] introduced a very interesting potential in the Dirac equation. Following the original Dirac procedure they introduce a system whose Dirac Hamiltonian is linear in both $\mathbf{p}$ and $\mathbf{r}$; consequently, its nonrelativistic approximation is quadratic in $\mathbf{r}$ as in the case of the harmonic oscillator. This property is in fact the origin of the name Dirac oscillator. The system is then, excepting for a strong spin-orbit coupling term, the square root of the harmonic oscillator in the same sense as the Dirac equation is the square root of the Klein-Gordon equation.

The Hamiltonian of the system is obtained by introducing in a nonminimal way the external potential with the substitution

$$\mathbf{p} \rightarrow \mathbf{p} - im\omega\beta\mathbf{r}\,, \tag{1}$$

where m is the mass of the particle and ω is the oscillator frequency. We follow the usual conventions concerning the Dirac matrices.[2] The Dirac equation for the system is then ($\hbar = c = 1$)

$$i(\partial\psi/\partial t) = H\psi = [\boldsymbol{\alpha}\cdot(\mathbf{p} - im\omega\beta\mathbf{r}) + m\beta]\psi\,. \tag{2}$$

The interaction introduced by Eq. (1) has been shown[3] to correspond to an anomalous magnetic interaction. In fact, if we select a frame-dependent vector[4]

$$u^\mu = m\omega(1,\mathbf{0})\,, \tag{3}$$

then the interaction term in Eq. (2) can be put in the form

$$im\omega\boldsymbol{\alpha}\cdot\mathbf{r} = \sigma^{\mu\nu}x_\mu u_\nu\,; \tag{4}$$

consequently, Eq. (2) can be written in the manifestly covariant form

$$[\gamma^\mu p_\mu - m + (ke/4m)\sigma^{\mu\nu}F_{\mu\nu}]\psi = 0\,, \tag{5}$$

with

$$F^{\mu\nu} = (u^\mu x^\nu - u^\nu x^\mu)\,, \tag{6}$$

and $k = 2m/e$ is the anomalous magnetic moment for the Dirac oscillator. The electromagnetic potential can be cast in the form

$$A_\mu = \tfrac{1}{2}[x_\mu(u\cdot x) - \tfrac{1}{2}x^2 u_\mu]\,; \tag{7}$$

but, taking advantage of the gauge invariance of the electromagnetic interaction and selecting the gauge $\lambda = \frac{1}{4}x^2(u\cdot x)$, we can write Eq. (7) in the equivalent forms

$$A_\mu^\pm = A_\mu \pm \partial_\mu\lambda = \begin{cases} \tfrac{1}{2}(u\cdot x)x_\mu\,, \\ -\tfrac{1}{2}x^2 u_\mu\,. \end{cases} \tag{8}$$

For either selection the electromagnetic field takes the form $\mathbf{E} = -m\omega\mathbf{r}$ and $\mathbf{B} = \mathbf{0}$, expressions which are, of course, frame dependent.[4] In this way we have shown that the interaction occurs as if it were produced by an infinite sphere carrying a uniform charge-density distribution, resulting in a linearly growing electric field. The interaction term introduced by Eq. (1) is interesting in its own right, but also has applications in quantum chromodynamics (QCD). For example, if we think of the construction of quark-confinement models, the linearly growing interaction characteristic of the system can be regarded as an effective chromoelectric field, at least if we think that the color-field lines are constrained to strings of constant volume instead of constrained to constant cross section, as is usually supposed.[3] Also, it can be useful for estimating the quark masses.[5] These considerations make the Dirac oscillator important to QCD models, as we will discuss in a forthcoming paper.[6]

Moshinsky and Szczepaniak have investigated the positive-energy part of the spectrum;[1] we proceed now to give the full solution for the eigenstates of the Dirac oscillator. We found the components of the Dirac wave function to be of the same form as the eigenfunctions of the three-dimensional nonrelativistic harmonic oscillator; hence the name *oscillator* applied to this system is also justified at this level. From the commutation relations

$$[\mathbf{L},H] = i(\boldsymbol{\alpha}\times\mathbf{p}) - m\omega(\mathbf{r}\times\boldsymbol{\alpha})\beta\,, \tag{9}$$

and

$$[\sigma/2,H]=-i(\boldsymbol{\alpha}\times\mathbf{p})+m\omega(\mathbf{r}\times\boldsymbol{\alpha})\beta\,, \qquad (10)$$

it immediately follows that the total angular momentum is conserved in our system. From Eq. (2) or, equivalently, from Eq. (5) we see that parity is a good quantum number. The sum rules of angular momentum imply that $j=l\pm\frac{1}{2}$, where j is the total angular momentum quantum number and l is the orbital one. The parity of the energy eigenfunction is given by $(-1)^l$, we therefore define

$$\epsilon=\begin{cases}+1 & \text{if parity is } (-1)^{j+1/2},\\ -1 & \text{if parity is } (-1)^{j-1/2};\end{cases} \qquad (11)$$

in both cases we have $l=j+\epsilon/2$. We write the solutions to Eq. (2) as

$$\Psi(\mathbf{r},t)=\frac{1}{r}\begin{pmatrix}F(r)\mathcal{Y}_l(\theta,\varphi)\\ iG(r)\mathcal{Y}_{l'}(\theta,\varphi)\end{pmatrix}\exp(-iEt)\,, \qquad (12)$$

where $\mathcal{Y}_l$ and $\mathcal{Y}_{l'}$ are spinor spherical harmonics of order l and l', respectively. Here, $l'=j-\epsilon/2$ as $\mathcal{Y}_{l'}$ must be of opposite parity to $\mathcal{Y}_l$.

The large radial component $F(r)$ and the small radial component $G(r)$ of the Dirac wave function are then solutions to the following coupled differential equations:

$$\{-d/dr+[\epsilon(j+\tfrac{1}{2})+m\omega r^2]/r\}G(r)=(E-m)F(r)\,, \qquad (13a)$$

and

$$\{+d/dr+[\epsilon(j+\tfrac{1}{2})+m\omega r^2]/r\}F(r)=(E+m)G(r)\,. \qquad (13b)$$

It is easy to see that the solutions to this system of equations are given by

$$F_{n,l}(r)=A[(m\omega)^{1/2}r]^{l+1}\exp(-m\omega r^2/2)\times L_n^{l+1/2}(m\omega r^2)\,, \qquad (14a)$$

and

$$G_{n,l'}(r)=A[(m\omega)^{1/2}r]^{l'+1}\exp(-m\omega r^2/2)\times L_n^{l'+1/2}(m\omega r^2)\,, \qquad (14b)$$

where $L_k^{\xi}(x)$ is an associated Laguerre polynomial, and $N=2n+l$, $n=0,1,2,\ldots$, is the principal quantum number.[7] The normalization constants are

$$A=\left(\frac{m\omega}{\pi}\right)^{1/4}\left(\frac{n!2^{n+l+3/2-\epsilon/2}}{(2n+2l+1-2\epsilon)!!}\right)^{1/2}\times[(n+l+1-\epsilon/2)^3+(n+l-\epsilon/2)^2]^{-1/2}\,. \qquad (14c)$$

The energy spectrum[1] can be obtained from

$$E^2-m^2=m\omega[2(N+1)+\epsilon(2j+1)] \qquad (15a)$$

for the positive-energy states, and from

$$E^2-m^2=m\omega[2(N+2)+\epsilon(2j+1)] \qquad (15b)$$

for the negative-energy states. Notice that, as we have already said, both $F(r)$ and $G(r)$ are of the same form as the radial eigenfunctions for the 3D nonrelativistic harmonic oscillator. It is important to point out here that the complete energy spectrum is given by both Eqs. (15a) *and* (15b). We are proceeding here in a similar way to the free case; that is, we must include the complete spectrum if we want to avoid problems such as the Klein paradox.[8] We notice also that the energy spectrum of the system is such that the term E^2-m^2 is the same, except for a constant $2m\omega$, for both the large component F and the small component G. Later it will be clear that this property is related to the hidden supersymmetry of the system. As expected from the O(3) symmetry of the problem, the spectrum is degenerate in j_z. But, as is readily apparent from Eqs. (15a) and (15b), there exists extra degeneracies, already noticed by Moshinsky and Szczepaniak.[1] First, for $\epsilon=-1$, all the states with $(N\pm s,\ j\pm s)$, where s is an integer, have the same energy; second, for $\epsilon=+1$, the states with $(N\pm s,\ j\mp s)$ also have the same energy. It will be shown in a forthcoming paper[9] that the symmetry algebra explaining this degeneracy is $SO(3,1)\oplus SO(4)$.

Let us address now the problem of the hidden supersymmetry. If we set

$$\Phi=\begin{pmatrix}F\\ G\end{pmatrix}, \qquad (16)$$

then we can construct an equivalent supersymmetric Hamiltonian[10-12] H_S acting only on the radial part of the eigenfunctions,

$$H_S\Phi=(E^2-m^2)\Phi\,. \qquad (17)$$

Here, we define

$$H_S=\{Q,Q^\dagger\}=-\frac{d^2}{dr^2}+\frac{[\epsilon(j+\frac{1}{2})+r^2]^2}{r^2}-\left(1-\frac{\epsilon(j+\frac{1}{2})}{r^2}\right)\sigma_3\,, \qquad (18)$$

where Q and $Q^\dagger$ are fermionic operators, the generators of the supersymmetry transformations,[13,14] defined as

$$Q=(p+iU')\psi \text{ and } Q^\dagger=(p-iU')\psi^\dagger\,, \qquad (19)$$

with

$$p=(1/i)\partial_r,\quad \psi=\sigma_1+i\sigma_2=\sigma^+\,, \qquad (20)$$

where the superpotential $U(r)$ is given by

$$U(r)=\epsilon(j+\tfrac{1}{2})\ln(r)+\tfrac{1}{2}m\omega^2r^2\,, \qquad (21)$$

and we take the usual conventions concerning the Pauli matrices. Therefore, we see that the large radial component $F(r)$ is a solution of the fermionic sector of H_S and is the supersymmetric partner of $G(r)$. This means that the energy spectrum of H_S, which is defined in Eq. (17) as E^2-m^2, must be the same (with the corresponding shift for each sector) for both radial components except for the base state. That this is indeed the case can be seen in Eqs. (15). On the other hand, it follows from the solutions (14) (or from the topology of the superpotential[13]) that the supersymmetry remains unbroken.

The operators defined in Eqs. (18) and (19) define a superalgebra in the following way:[6,10,13,15]

$$\{Q,Q^\dagger\}=H_S,\quad [Q,\lambda]=-2\lambda Q,\quad [Q^\dagger,\lambda]=2Q^\dagger\lambda\,, \tag{22}$$

where $\lambda=\sigma_3 m$. From Eq. (19) we can show that λ anticommutes with Q and $Q^\dagger$ separately, and, as a consequence, it commutes with the supersymmetric Hamiltonian H_S, closing the algebra defined in Eq. (22). It is shown in Ref. 15 that this is precisely the condition of stability of the Dirac sea. When the aforementioned superalgebra is closed, it is possible to construct an exact Foldy-Wouthuysen transformation for the Dirac-oscillator Hamiltonian. This means that, regardless of the intensity of the coupling, the positive- and the negative-energy solutions never mix when the interaction is turned on, leaving the vacuum of the theory unchanged.

Thus we have shown that the Dirac oscillator is a system that behaves like an odd potential, it is exactly soluble, it conserves angular momentum, and it is such that the special characteristic of its energy spectrum, namely, that the positive- and the negative-energy states never mix, is intimately related with a hidden supersymmetry produced by the "anomalous momentum" $im\omega\beta\mathbf{r}$. These considerations and the solution described here can be important for the construction of heavy-quark models, as the interaction term appearing in the Hamiltonian of the Dirac oscillator can be regarded as a confining chromoelectric field in which the color-field lines are bundled in strings of constant volume.

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