Interactions and Excitations of Non-Abelian Vortices

M. G. Alford, Katherine Benson, Sidney Coleman, and John March-Russell
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

Frank Wilczek
Institute for Advanced Study, Princeton, New Jersey 08540
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We examine bosonic zero modes of vortices formed in the gauge breaking $G \rightarrow H$. For non-Abelian G, zero modes are generic. Their solutions depend on global symmetry structure. Vortices render the embedding $H \subset G$ space dependent, with a dynamically determined subgroup $\tilde{H}$ single valued. They Aharonov-Bohm scatter gauge bosons associated with multivalued generators. Alice strings $[H = O(2), \tilde{H} = \mathbb{Z}_2]$ attract charges and scatter SO(2) "photons," and a two-string system has zero modes with unlocalizable "Cheshire" charge. The resulting superconductivity has novel electrodynamics.

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Vortices, line defects containing gauge flux, occur generically in many theories, both in condensed-matter systems and as cosmic strings in grand unified theories. Such solitons are usually treated as classical backgrounds; however, in their interactions with quantum fields it is essential to consider the possible multivaluedness of unbroken symmetries, as well as any massless excitations ("zero modes") of the solitons. Multivalued symmetries lead to Aharonov-Bohm scattering of gauge bosons by vortices, while zero modes can induce such dramatic effects as cosmic-string superconductivity[1] and baryon-number violation.[2] Further, the relationship between the original theory and the induced lower-dimensional field theory describing the zero modes is interesting in its own right. In this paper we summarize the results of our investigation into the symmetries and zero modes of non-Abelian vortices. A more detailed account will be published elsewhere.[3]

(1) Global symmetries.—Consider a gauge group G broken to a subgroup H by a condensate $\langle\phi\rangle$. When the vacuum manifold G/H has a nontrivial fundamental group stable vortices can form, with $\langle\phi\rangle$ winding at spatial infinity. These strings can be characterized by the Wilson-line integral

$$U(\theta) = P\exp\left[\int_0^\theta \mathbf{A}\cdot d\mathbf{l}\right], \tag{1.1}$$

at infinite radius. This generates the condensate winding at spatial infinity:

$$\langle\phi(\theta)\rangle = U(\theta)\langle\phi(0)\rangle. \tag{1.2}$$

Thus $U(2\pi) = h \in H$, where H is the little group of the Higgs field $\langle\phi(0)\rangle$, since $\langle\phi\rangle$ must be single valued. The vortices that can form in this breaking are therefore specified by the possible values of $U(2\pi)$—the elements of H. Note that this classification is finer than specifying the topological class of the string, i.e., its element of $\pi_1(G/H)$.

Now consider the possibility, first noticed by Schwarz,[4] that the local unbroken symmetry group H cannot be globally extended, and that $\tilde{H}$ is the subgroup of globally well-defined symmetries (as for monopoles[5]). To illustrate this, consider a string with given $U(2\pi) \in H$. Since the Higgs condensate winds via (1.2), the embedding of H in G changes as θ varies, yielding a family of different, but isomorphic subgroups $H(\theta)$. At $\theta = 0$ choose a basis $\{S_a(0)\}$ for the Lie algebra $\mathfrak{L}(H)$ of H, where $a = 1, \ldots, \dim(H)$ and $\mathrm{tr} S_a^\dagger S_b = \delta_{ab}$. The Lie-algebra elements $S_a(\theta)$ that generate the local $H(\theta)$ are given by parallel transport:

$$S_a(\theta) = U(\theta) S_a(0) U^{-1}(\theta). \tag{1.3}$$

They generate a globally well-defined symmetry group only if they are all single valued; however, in general,

$$S_a(2\pi) = U(2\pi) S_a(0) U^{-1}(2\pi) = R_{ab}(U(2\pi)) S_b(0). \tag{1.4}$$

By a basis change on $\mathfrak{L}(H)$, R_{ab} can be diagonalized:

$$U(2\pi) S_a(0) U^{-1}(2\pi) = \lambda_a(U(2\pi)) S_a(0) \quad (\text{no sum}), \tag{1.5}$$

where $\lambda_a = \exp(2\pi i \xi_a)$. Thus a globally well-defined symmetry group $\tilde{H}$ of our string is generated by the $\lambda_a = +1$ subalgebra. In terms of group elements, $\tilde{H}$ is just the centralizer of $U(2\pi)$ within H:

$$\tilde{H} = \{h \in H \mid [U(2\pi), h] = 0\}. \tag{1.6}$$

This is a nontopological criterion—an element of the homotopy group $\pi_1(G/H)$ only tells us the disconnected component of H to which $U(2\pi)$ belongs, leaving many possible $U(2\pi)$ with different groups $\tilde{H}$.

To illustrate this, consider $G = SO(6)$ with a Higgs field ϕ transforming as the **20** (a symmetric, traceless, 6×6 matrix, so $\langle\phi\rangle \rightarrow g\langle\phi\rangle g^{-1}$, with g in the fundamental). If $\langle\phi\rangle$ acquires the vacuum expectation value $v\,\mathrm{diag}(1^3,-1^3)$, where the power denotes the multiplicity of the entry, then this condensate leaves unbroken an $SO(3)\times SO(3)$ subgroup of $SO(6)$ and a discrete $\mathbb{Z}_2$ transformation generated by $g_1=-\mathbb{1}_6$. So the complete little group of $\langle\phi\rangle$ is $H=SO(3)\times SO(3)\times\mathbb{Z}_2$. Here $\pi_1(G/H)=\mathbb{Z}_2$, so topologically stable strings can form, with $U(2\pi)$ an element of the disconnected component of H. Consider the following candidates for $U(2\pi)$: (1) g_1, (2) $g_2=\mathrm{diag}(1^2,-1^4)$, and (3) $g_3=-R_{12}(\beta)$, with $R_{12}(\beta)$ a rotation through β in the 1-2 plane, mediating between case (1) and case (2). Note that only g_1 commutes with all $SO(3)\times SO(3)$ transformations, so the true, globally defined, unbroken symmetry group $\tilde{H}$ in the presence of such a $\mathbb{Z}_2$ string depends on our choice: (1) If $U(2\pi)=g_1$, the centralizer of $U(2\pi)$ is all of $H=SO(3)\times SO(3)\times\mathbb{Z}_2$, and this is the globally defined symmetry group. (2) If $U(2\pi)=g_2$, $\tilde{H}=SO(2)_{12}\times SO(3)\times\mathbb{Z}_2$; generators T_{13} and T_{23} of rotations in the 1-3 and 2-3 plane are double valued. (3) More generally, if $U(2\pi)=g_3$, for $\beta\neq 0$ (mod 2π), $\tilde{H}=SO(2)_{12}\times SO(3)\times\mathbb{Z}_2$; generators $(1/\sqrt{2})(T_{13}\pm iT_{23})$ are multivalued, acquiring the phase $\exp(\mp i\beta)$ in circling the string.

These strings, although characterized by total flux within the same homotopy class, lead to different globally defined symmetry groups $\tilde{H}$. Which possibility is realized in a given model is a dynamical question, depending on details of the interactions.

(2) Zero-mode excitations.—Zero modes exist when a soliton is not invariant under a continuous unbroken symmetry of the Lagrangian. Using this criterion most strings have zero modes since the flux in the core is generally not invariant under H rotations. Indeed, generators of H but not $\tilde{H}$ always lead to nontrivial rotations.

We now construct the zero modes. Consider the Lagrangian

$$\mathcal{L}=\tfrac{1}{4}\,\mathrm{tr}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}(D_\mu\phi)^\dagger D^\mu\phi-V(\phi)\,, \quad (2.1)$$

where $A_\mu=A_\mu^a T^a$, $D_\mu\phi=\partial_\mu\phi-A_\mu\phi$, and T^a is an anti-Hermitian generator in the ϕ representation with $\mathrm{tr}(T^{a\dagger}T^b)=\delta^{ab}$. ϕ condenses, breaking the gauge group G (locally) to H. The field equations are

$$D^\mu D_\mu\phi=-2\frac{\partial V}{\partial(|\phi|^2)}\phi \text{ for } \phi\,, \quad (2.2)$$

$$D^\mu F_{\mu\nu}=\sum_a T^a(D_\nu\phi)^\dagger T^a\phi \text{ for } A_\mu\,. \quad (2.3)$$

They have as one solution a string along the z axis:

$$A_0=A_z=0,\ A_r=\bar{A}_r(r,\theta),\ A_\theta=\bar{A}_\theta(r,\theta)\,, \quad (2.4)$$

$$\bar{\phi}=W(\theta)\langle\phi(r)\rangle\,, \quad (2.5)$$

where $\langle\phi(r)\rangle$ approaches 0 at the origin and constant at infinity, $W(\theta)$ goes to $U(\theta)$ at infinity, and $\bar{A}_r,\bar{A}_\theta$ have the boundary conditions

$$\bar{A}_r=0,\ \bar{A}_\theta=0 \text{ for } r=0\,, \quad (2.6)$$

$$\bar{A}_r\to 0,\ \bar{A}_\theta\to -(n/r)T \text{ for } r\to\infty\,. \quad (2.7)$$

Condition (2.6) excludes singular charge or current distributions at the origin; condition (2.7) specifies the winding number n and a generator $T\in\mathfrak{L}(G)$.

Generators S_a acting nontrivially on this flux condensate (or alternatively, on some noninvariant scalar condensate) induce zero modes whose form we seek. Since we want solutions whose energies vanish when there is no t and z dependence, a natural prescription is (z,t) modulation of a gauge transformation in the x-y plane:

$$\phi=\Omega\bar{\phi},\ A_\mu=\Omega\bar{A}_\mu\Omega^{-1}+\delta_\mu^i(\partial_i\Omega)\Omega^{-1}\,, \quad (2.8)$$

where $\Omega=\Omega(\mathbf{x},t)\in H$ and i,j label x,y while α,β label z,t. Thus A_0 and A_z remain zero while A_x and A_y are gauge transformed. Gauge transforming by Ω^{-1} gives the alternative formulation

$$\phi=\bar{\phi},\ A_\mu=\bar{A}_\mu-\delta_\mu^\alpha(\partial_\alpha\Omega)\Omega^{-1}\,, \quad (2.9)$$

in which it is clear that the energies can depend only on the z and t derivatives of Ω. We solve for Ω in the linear approximation: For Ω generated by $\omega(\mathbf{x},t)\in\mathfrak{L}(H)$, we retain only terms linear in ω, approximating $\partial_\mu\Omega\,\Omega^{-1}$ by $\partial_\mu\omega$. Thus, under the *Ansatz* (2.9), the string's field strength is augmented by $\delta F_{\alpha\beta}=0$, $\delta F_{i\alpha}=-\bar{D}_i(\partial_\alpha\omega)$, and $\delta F_{ij}=0$, where $\bar{D}_i\omega=\partial_i\omega-[\bar{A}_i,\omega]$. Linearizing the gauge field equations (2.3), we obtain

$$\bar{D}^i\bar{D}_i\partial_\alpha\omega=-\sum_a T^a(\partial_\alpha\omega\bar{\phi})^\dagger T^a\bar{\phi} \text{ for } \nu=\alpha\,, \quad (2.10)$$

$$\bar{D}_i\partial^\alpha\partial_\alpha\omega=0 \text{ for } \nu=i\,, \quad (2.11)$$

which, as linear equations, can be solved by the separation of variables $\omega=\eta(z,t)\alpha(r,\theta)$, where $\alpha(r,\theta)$ is Lie-algebra valued. For (2.11) this gives [since physical excitations have $\bar{D}_i\alpha(r,\theta)\not\equiv 0$]

$$\partial^\alpha\partial_\alpha\eta(z,t)=0\,, \quad (2.12)$$

so we have massless modes propagating along the string at the speed of light. These modes automatically obey Eq. (2.2) for ϕ.

The (r,θ) dependence of the zero modes is determined by (2.10), giving

$$\bar{D}^i\bar{D}_i\alpha=-\sum_a T^a(\alpha\bar{\phi})^\dagger T^a\bar{\phi}\,. \quad (2.13)$$

Such a Laplace-type equation always gives a unique solution when a single-valued $\alpha(r,\theta)$ is specified at spatial infinity. To specify the asymptotic behavior of $\alpha(r,\theta)$, we consider Eq. (2.13) as $r\to\infty$. Here the right-hand side vanishes and $\bar{A}_\mu$ assumes the form (2.7), so $\bar{D}_r\alpha=\partial_r\alpha$ and $(1/r)\bar{D}_\theta\alpha=(1/r)\partial_\theta\alpha+(n/r)[T,\alpha]$. Thus we may write (2.13) asymptotically as

$$(1/r)\partial_r(r\partial_r\alpha)+(1/r^2)\Delta_\theta^2\alpha=0\,, \quad (2.14)$$

where Δ_θ, defined by $\Delta_\theta \alpha = \partial_\theta \alpha + n[T,\alpha]$, has no r dependence. Therefore, Δ_θ commutes with $\bar{D}^i \bar{D}_i$ at spatial infinity and we may expand α in Δ_θ eigenstates:

$$\Delta_\theta \alpha_s = is\alpha_s, \quad s \in \mathbb{R}. \tag{2.15}$$

The explicit solution is

$$\alpha_s(\infty,\theta) = \exp(is\theta)\exp(-n\theta T)\alpha_s(\infty,0)\exp(n\theta T).$$

Recalling the conjugated generators of H, $S_a(\theta) = \exp(-n\theta T)S_a(0)\exp(n\theta T)$, we see that for α proportional to a single-valued generator $\tilde{S}_a(\theta)$, $s \in \mathbb{Z}$. However, for α proportional to a multivalued generator with $S(2\pi) = \exp(2\pi i \xi)S(0)$, single valuedness of α implies that $s \in \mathbb{Z} - \xi$.

By (2.14), the asymptotic r dependence of α_s is $r^{\pm s}$ for nonzero s and $\ln r + \text{const}$ for $s=0$. We now multiply the full zero-mode equation (2.13) for such an eigenstate by $\alpha_s^\dagger$, integrate over (r,θ), and take the trace to obtain (after integrating by parts)

$$\mathrm{tr}\int_0^{2\pi} d\theta\, \alpha_s^\dagger r \partial_r \alpha_s \Big|_{r=0}^{\infty} = \int_0^\infty \int_0^{2\pi} r\,dr\,d\theta \left[\mathrm{tr}(\bar{D}_r \alpha_s)^\dagger (\bar{D}_r \alpha_s) + \frac{1}{r^2}\mathrm{tr}(\bar{D}_\theta \alpha_s)^\dagger (\bar{D}_\theta \alpha_s) + |\alpha_s \bar{\phi}|^2 \right] \geq 0, \tag{2.16}$$

where on the left-hand side we have used the behavior (2.7) of $\bar{A}_r$ at $r=0$ and $r \to \infty$ to replace $\bar{D}_r$ by ∂_r. Since for a nontrivial zero mode the right-hand side of (2.16) is strictly positive, and α is bounded at the origin, the zero mode must induce a positive surface term at spatial infinity. Therefore α must always contain some admixture of the radially diverging [i.e., $\sim r^{|s|}$ or $\sim \ln(r)$] solution of (2.13). The energy of the mode is

$$\begin{aligned} E &= \int_0^R r\,dr\,\mathrm{tr}(\delta F_{ia})^\dagger(\delta F_{ia}) \\ &= \int_0^R r\,dr (\partial_a \eta)^2 \mathrm{tr}(\bar{D}_i \alpha)^\dagger (\bar{D}_i \alpha) \quad (R \to \infty). \end{aligned} \tag{2.17}$$

The $(\partial_a \eta)^2$ factor is just λ^{-2} for a zero mode of z wavelength λ, so

$$\begin{aligned} &E \sim \lambda^{-2} R^{2|s|} \text{ for } s>0, \\ &E \sim \lambda^{-2} \ln R \text{ for } s=0. \end{aligned} \tag{2.18}$$

Clearly for fixed R the energy vanishes in the long-wavelength limit, as expected for zero modes. However, for fixed λ the energy diverges as $R \to \infty$. Note that the mildest energy divergence is the logarithmic one for $s=0$ solutions, which only exist for $S_a \in \mathfrak{L}(\tilde{H})$. This logarithmic divergence is not surprising; it arises whenever charged particles travel at the speed of light, due to Lorentz contraction of the field into a pancake with $E_r \sim 1/r$. Conversely, any solution associated with a generator in H but not $\tilde{H}$ has polynomially divergent energy. Thus the energy of these modes depends critically and in a very unusual way upon the large-scale structure of the string.

(3) Alice electrodynamics.— We have now described the global nonexistence of locally valid symmetries as a formal possibility that generically arises when H is non-Abelian. Physically this induces different amplitudes for the Aharonov-Bohm scattering of particles within a single H multiplet off the string. This occurs even when H is discrete.[6] More dramatic consequences arise when H contains a continuous part. We now consider a simple example of this type, the Alice string.[4]

The Alice string occurs in a model where SO(3) (generated by T_1, T_2, T_3 with $\mathrm{tr}T_a^\dagger T_b = 2\delta_{ab}$) is broken to O(2) by a Higgs ϕ transforming as a 5 with $\langle \phi(\theta=0)\rangle = v\,\mathrm{diag}(1,1,-2)$. At $\theta=0$ the unbroken O(2) consists of a U(1) generated by T_3 and a discrete element X corresponding to rotation by π about the x axis. For a string with $U(2\pi)=X$ the Higgs condensate becomes angle dependent, $\langle\phi(r,\theta)\rangle = \exp(\theta T_1/2)\langle\phi(r,0)\rangle \times \exp(-\theta T_1/2)$, with the unbroken U(1) generated by $S_3(\theta)$, where $S_a(\theta) = \exp(\theta T_1/2)T_a\exp(-\theta T_1/2)$. Thus $S_2(\theta)$ and $S_3(\theta)$ are double valued, and there is no globally defined unbroken continuous symmetry, only a double-valued U(1) generated by $S_3(\theta)$. This leads to three striking consequences: (1) Charged particles are attracted to the string. (2) The massive S_2 gauge boson and the massless "photon" associated with S_3 both undergo Aharonov-Bohm scattering off the string. (3) A pair of strings can carry a charged zero mode, but the charge cannot be localized either to the string or to the space between them ("Cheshire charge").

We now briefly explain these features. A detailed treatment will be presented elsewhere.[3] Writing $A_\mu = A_\mu^{(a)} S_a$, etc., we find that outside the string core the Lagrangian is

$$\begin{aligned} \mathcal{L} = -\tfrac{1}{4} F_{\mu\nu}^{(a)} F^{(a)\mu\nu} &+ 9g^2 v^2 (A_\mu^{(1)} A^{(1)\mu} + A_\mu^{(2)} A^{(2)\mu}) \\ &+ \text{interactions}. \end{aligned} \tag{3.1}$$

Thus there is one massless gauge particle ($A_\mu^{(3)}$) associated with the locally unbroken U(1) symmetry, and two massive ones associated with the broken generators of SO(3). Note that S_2, S_3, $A^{(2)}$, and $A^{(3)}$ are all double-valued functions of θ, so that A_μ is single valued. For a U(1)-charged particle in the presence of the string, it can be shown that (on the first Riemann sheet) $A_\mu^{(3)}$ obeys the Maxwell equations for the field of a charged particle at $x=R$, $y=z=0$, with the branch cut along the $y=0$, $x<0$ plane replaced by conducting-plate boundary conditions. A real conducting plate induces an opposite charge on the plate, attracting the original particle. Since the U(1) field for the Alice string is identical, charge is attracted to the string [property (1) above].

Property (2) is seen by expanding an incoming plane wave of $A_\mu^{(a)}$ in solutions of the equations of motion.

The components of $A^{(a)}$ obey the Klein-Gordon equation, whose solutions take the form $J_w(kr)\exp(iw\theta)$. The strength of Aharonov-Bohm scattering is determined by the order w of the Bessel function J_w, specifically $\sigma \propto \sin^2(w\pi)$.[7] For the single-valued field $A_\mu^{(1)}$, $w \in \mathbb{Z}$, so there is no Aharonov-Bohm scattering. However, for the other two bosons (including the photon $A_\mu^{(3)}$) $w \in \mathbb{Z}+\frac{1}{2}$, to obtain double-valued functions, so there is maximal scattering.

To discuss property (3) for a pair of strings at $(x,y) = (0,0)$ and $(D,0)$ we use singular gauge, in which the background field $\overline{A}_\mu$ is zero except on the plane joining the two strings. This plane is then the branch cut, and $S_3(x,y)$ and the electric field E_i both change sign on crossing it $[F_{0i}(x,y) = E_i(x,y)S_3(x,y)]$. Outside the string cores, the zero-mode equation of motion (2.13) yields the source-free Gauss law, $\partial_i E_i = 0$. E_i must change sign on circling one string, but is single valued on circling both strings. Thus there is a charged $s=0$ zero mode for the two-string system, for which $E_r(r,\theta) = Q/r$ for large r. Gaussian surfaces enclosing no string contain no net charge; however, no meaning can be assigned to those enclosing a single string because on them E_i is double valued. Thus we cannot localize the charge to a single string nor to the intermediate region using Gaussian surfaces. To obtain more information we could calculate $E_i(x,y)$. It can be shown, by reflection symmetry in the x and y axes, that the branch cut is equivalent to a conducting plate with charge Q. Solving the electrostatic equation with this boundary condition reveals that $E_y(x,0) = Q/[x(D-x)]^{1/2}$ for $0 < x < D$. So as we take one string to infinity the charge is not localized near either string.

Thus although the generic non-Abelian vortex pair (or loop) necessarily supports a form of superconductivity, in general the associated electrodynamics will be quite different from that of a superconducting wire.

The principal physical relevance of zero modes lies in the superconductivity that they generate,[1] and, further, their ability to absorb charges onto the string.[2,3] Indeed, we may understand the necessary existence of some zero modes by considering a loop of string with flux $U(2\pi)$: All the symmetries H can then be defined globally at infinity, along with the associated charges. However, if a particle slowly threads the loop it returns conjugated by $U(2\pi)$, and this generally alters the quantum numbers associated with multivalued generators. Since these are nevertheless good quantum numbers, there must be low-energy excitations of the string capable of absorbing the deficit. Further, this argument shows that there can be *magnetically* charged excitations of vortex loops. For instance, in the Alice-string example we could choose to take a monopole through the loop. These magnetic modes are discussed in Ref. 3.

The inelastic-scattering process excites a finite-energy superposition of modes on the string, whose charge per unit length drops to zero as their electric field spreads out from the string. This enables charge to be deposited at an energy cost that is, in principle, arbitrarily small, although there may exist energy barriers at intermediate stages. The details of this process are currently under investigation.

After the completion of this work we received a paper by Preskill and Krauss[8] in which symmetries in the presence of a string and Cheshire charge are discussed.

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