# New Exact Solution for the Exterior Gravitational Field of a Spinning Mass

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An exact asymptotically flat solution of the vacuum Einstein equations representing the exterior gravitational field of a stationary axisymmetric mass with an arbitrary mass-multipole structure is presented.

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To describe correctly the gravitational field of a spinning mass, one needs an exact asymptotically flat solution of the vacuum Einstein equations possessing an arbitrary multipole structure provided by two independent infinite sets of arbitrary parameters. Neither the well-known Kerr metric[1] nor recently found metrics[2,3] with the Schwarzschild static limit solve this problem because of a special relationship between their multipole moments (see, e.g., Ref. 4). Another stationary solution obtained by Quevedo and Mashhoon,[5,6] though it takes into account static deformations of a star, does not describe all possible deformations due to rotation. In this Letter, we present in an explicit and very compact form a metric which contains two infinite, yet dependent, sets of arbitrary parameters $\alpha_n$ and $\beta_n$, and accounts for both static and stationary deformations of an axisymmetric mass, its angular momentum multipole moments (except for the total angular momentum) being dependent upon the choice of the mass-multipole moments. In the absence of rotation, our solution reduces to a new general static axisymmetric asymptotically flat vacuum metric in which the gravitational multipoles are superimposed upon the Schwarzschild solution.

As is known, the general stationary axisymmetric vacuum problem reduces to the Ernst equation[7]

$$(\epsilon+\epsilon^*)\Delta\epsilon=2(\nabla\epsilon)^2 \tag{1}$$

for a complex Ernst potential $\epsilon$ which determines the functions $f$, $\gamma$, and $\omega$ in the line element

$$ds^2=k^2f^{-1}\{e^{2\gamma}(x^2-y^2)[dx^2/(x^2-1)+dy^2/(1-y^2)]+(x^2-1)(1-y^2)d\varphi^2\}-f(dt-\omega\, d\varphi)^2, \tag{2}$$

where $k$ is a real constant and $(x,y)$ are prolate spheroidal coordinates.

Using the nonlinear superposition technique[8-12] we were able to derive the following solution of Eq. (1) and the corresponding metric functions $f$, $\gamma$, and $\omega$:

$$\begin{aligned}
&\epsilon=A_-/A_+,\quad f=2p(x^2-1)\exp(2\psi)A/B,\\
&\exp(2\gamma)=\exp(2\gamma')A/(x-y)^8,\quad \omega=2kq+kqp^{-1}(-4y+\hat{p}-C/A),\\
&A_\mp=(p+1)(x\mp1)[(x-y)^4\mp iq(x\mp1)(1\pm y)(x^2-1)\exp(2a)]\exp(\pm\psi)\\
&\qquad+iq(x\pm1)[(x-y)^4\pm iq(x\pm1)(1\mp y)(x^2-1)\exp(2a)]\exp(\mp\psi),\\
&A=(x-y)^8-q^2(1-y^2)(x^2-1)^3\exp(4a),\\
&B=(p+1)(x+1)^2[(x-y)^8+q^2(x+1)^2(1-y)^2(x^2-1)^2\exp(4a)]\\
&\qquad+(p-1)(x-1)^2[(x-y)^8+q^2(x-1)^2(1+y)^2(x^2-1)^2\exp(4a)]\exp(4\psi)\\
&\qquad+4q^2(x^2-1)^2(x+y)^5\exp(2\psi+2a),
\end{aligned} \tag{3}$$

where $p$ and $q$ are real constants subject to $p^2-q^2=1$, while the functions $\psi$, $\gamma'$, $a$, and $\hat{p}$ are given by the expressions

$$\psi=\sum_{n=1}^{\infty}(\alpha_n+q\beta_n)(x+y)^{-n-1}P_n((xy+1)/(x+y)), \tag{4}$$

$$a=\sum_{n=1}^{\infty}\sum_{l=0}^{n}(\alpha_n+q\beta_n)2^{-n-1}[2^l(x-y)(x+y)^{-l-1}P_l-1], \tag{5}$$

$$\begin{aligned}\gamma'=&\tfrac{1}{2}\ln[(x^2-1)/(x^2-y^2)]\\&+\sum_{n=1}^{\infty}(\alpha_n+q\beta_n)\left[(x+y)^{-n-1}P_{n+1}+2^{-n-1}\sum_{l=0}^{n}[2^l(x-y)(x+y)^{-l-1}P_l-1]\right]\\&+\sum_{m,n=1}^{\infty}(\alpha_m+q\beta_m)(\alpha_n+q\beta_n)(m+1)(n+1)(P_{m+1}P_{n+1}-P_mP_n)/(m+n+2)(x+y)^{m+n+2},\end{aligned} \tag{6}$$

$$\hat{p}=2\sum_{n=1}^{\infty}(\alpha_n+q\beta_n)[(xy+1)(x+y)^{-1}P_n-P_{n-1}]/(x+y)^{n+1}. \tag{7}$$

Here $\alpha_n$ and $\beta_n$ are real constants; $P_n$ are the Legendre polynomials of the first kind, all of them being of the argument $(xy+1)/(x+y)$. Note, that the summation in the formulas (4)-(7) can be cut off at any desired value of $n$.

Relations (3)-(7) fully determine the new stationary metric. By calculating its Geroch-Hansen multipole moments[13,14] it is straightforward to see that the metric is asymptotically flat (the angular momentum monopole moment is equal to zero), and its total mass $M$, angular momentum $J$, and quadrupole moment $Q$ are given by

$$M=p,\quad J=q(p^2+2)/p,\qquad Q=-pq^2-(\alpha_2+q\beta_2)/p \tag{8}$$

($k=1$ and $\alpha_1=\beta_1=0$ are assumed for simplicity). In general, the mass-multipole moments $M_i$ and the angular momentum multipole moments $J_i$ contain the parameters $\alpha_i,\alpha_{i-1},\ldots$ and $\beta_i,\beta_{i-1},\ldots$, which define, respectively, the static and stationary deformations of an axisymmetric mass. Indeed, transition to the static case in our solution is achieved by putting $q=0$ in (3)-(6); then the resulting function

$$f=(x-1)(x+1)^{-1}\times\exp\left[\sum_{n=1}^{\infty}\alpha_n(x+y)^{-n-1}P_n((xy+1)/(x+y))\right] \tag{9}$$

together with $\gamma'$ arising from (6) fully determine the new static vacuum general axisymmetric solution describing a nonrotating mass with the entire set of arbitrary relativistic multipole moments, $\alpha_n$ defining up to a constant factor the $2^n$-pole Newtonian moment. On the other hand, when $\alpha_n=0$, $\beta_n\neq0$, the relations (3)-(7) determine stationary solutions with the Schwarzschild static limit, so that $\beta_n$ appear and survive only in a stationary case, defining the deformations of a mass due to rotation. It should be noted, however, that since the parameters $\alpha_n$ and $\beta_n$, separating static deformations of a mass from those due to rotation, are introduced into the solution by the combination $\alpha_n+\alpha\beta_n$, they can determine only one infinite set of arbitrary multipole moments ($M_i$ in our case), while the other (the angular momentum multipole moments $J_i$, $i\geq2$) is dependent upon the choice of $M_i$.

The simplest possible stationary metric contained in (3)-(7) is the one derived in Ref. 3, and it corresponds to the case $\alpha_n=\beta_n=0$. In the general case, when $\alpha_n\neq0$, $\beta_n\neq0$, the formulas (3)-(7) describe the exterior field of a stationary rotating deformed mass possessing an arbitrary mass-multipole structure. Our solution has an event horizon provided by the null hypersurface $x=1$, which is regular everywhere except at poles $y=\pm1$; the proper area of the horizon is given by

$$S=8\pi M^2(p+1)p^{-3}\exp\left[-\sum_{n=1}^{\infty}(\alpha_n+q\beta_n)/2^n\right], \tag{10}$$

and it reduces, at $\alpha_n=\beta_n=0$, to the expression obtained for the solution in Ref. 3, and at $q=0$, $\alpha_n=0$ to that for the Schwarzschild metric. The reported solution is algebraically general; however, it degenerates to Petrov type $D$ at the symmetry axis ($y=\pm1$) and at $q=\alpha_n=0$. The formulas obtained are easy for practical use: One only needs to choose the required number of arbitrary mass-multipole moments, to fix the index $n$ in the relations (4)-(7), and lastly to substitute $\psi$, $a$, $\gamma'$, and $\hat{p}$ into (3).

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