Wave-Kinetic Formulation of Incoherent Linear Mode Conversion

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We incorporate linear mode conversion in wave-kinetic theory when the incoming waves are incoherent, as occurs when the wave sources are turbulent processes or thermal emission. Mode conversion takes place when the rays of either wave mode cross the *mode-conversion manifold* in the ray phase space. Jump conditions are derived which match the incoming and outgoing wave actions across the mode-conversion manifold, leading to a pair of coupled wave-kinetic equations. We also discuss how incoherent linear mode conversion acts as a source of wave entropy.

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Linear mode conversion is an important process that is often encountered in nature. Much progress has been made recently on this fundamental problem (see, e.g., Refs. 1-7). In particular, it is now possible to solve the mode-conversion problem in a covariant manner in arbitrary geometries. In this paper we report an extension of these techniques which allows mode conversion to be incorporated into a wave-kinetic formulation. An important result is the discovery that incoherent mode conversion acts as a source of wave entropy.

We first review the basic developments on mode conversion and wave kinetics reported in Refs. 3, 5, 8, and 9. Then we discuss how to incorporate mode-conversion effects into wave kinetics.

Consider a weakly inhomogeneous, possibly time-varying medium which contains a mode-conversion region. The Friedland reduction technique[3] can be used to recast the general N-component integral wave equation into the simpler 2×2 form:

$$\int dx''\,\mathbf{D}(x',x'')\cdot \mathbf{Z}(x'')=0\,, \tag{1}$$

where the kernel $\mathbf{D}$ is a 2×2 matrix, $\mathbf{Z}$ is a two-component field which contains the relevant pair of polarizations, and x^μ is the space-time position $(\mathbf{x},t)$. Using the symbol calculus[8,9] Eq. (1) becomes $\mathbf{D}(k\rightarrow -i\partial/\partial x,x)\cdot\mathbf{Z}(x)=0$. Explicitly,

$$\mathbf{D}(k,x)\cdot\mathbf{Z}(x)=\begin{bmatrix} D_a(k,x) & \eta'(k,x) \\ \eta'^*(k,x) & D_b(k,x)\end{bmatrix}\begin{bmatrix} Z_a(x) \\ Z_b(x)\end{bmatrix}=0\,. \tag{2}$$

Here η' is the (small) coupling which mediates the mode conversion and k_μ is the four-vector $(\mathbf{k},-\omega)$. We assume that the dispersion matrix $\mathbf{D}$ is Hermitian, which is tantamount to neglecting dissipation. Consider first regions where *either* $D_a(k,x)=0$ *or* $D_b(k,x)=0$, but not both. We signify this by $D_n(k,x)=0$ for $n=a$ or b. At a given space-time point, x, this implies a relation among the components of the four-vector k_μ:

$$D_n(k,x)=0 \Longrightarrow \omega=\Omega_n(\mathbf{k},x)\,; \tag{3}$$

i.e., the mode n must satisfy the *dispersion relation* given by Eq. (3). In general, for most space-time regions, if $\mathbf{k}$ and ω satisfy one of the dispersion relations, they will not satisfy the other. However, in those regions where both dispersion relations are satisfied the two modes interact and linear conversion occurs. This is brought out clearly by considering the eight-dimensional *ray phase space* (k,x). The condition $D_n(k,x)=0$ defines a seven-dimensional surface called the *dispersion manifold* for mode n. We assume that the two dispersion manifolds intersect transversely. This implies that their intersection is six dimensional. The *mode-conversion manifold* we signify by $M_6\equiv\{(k,x)\colon D_a(k,x)=0,\ D_b(k,x)=0\}$.

In this work we are interested in short-wavelength solutions of Eq. (2). By "short" we mean that the eikonal approximation is valid for each mode *away from the mode-conversion manifold* M_6. Accordingly, there is a wave-kinetic equation for each wave mode,[8-10] which is a statement of action conservation following the respective ray trajectories. On the surface $D_n=0$ the rays of mode n obey $dx^\mu/d\sigma_n=-\partial D_n/\partial k_\mu$ and $dk_\mu/d\sigma_n=\partial D_n/\partial x^\mu$. Thus, $D_n(k,x)$ acts as the Hamiltonian for rays of mode n, and σ_n is the ray-orbit parameter related to t via $dt/d\sigma_n=\partial D_n/\partial\omega\equiv\partial_\omega D_n$. For definiteness we assume $\partial_\omega D_n>0$. (More general cases will be considered in a longer paper.) The evolution (following rays of mode n) of any phase-space function, $F(k,x)$, is given by $dF/d\sigma_n=\{D_n,F\}_8$, where $\{,\}_8$ denotes the canonical Poisson bracket on eight-dimensional phase space (k,x):

$$\{F,G\}_8\equiv\frac{\partial F}{\partial x^\mu}\frac{\partial G}{\partial k_\mu}-\frac{\partial F}{\partial k_\mu}\frac{\partial G}{\partial x^\mu}\,.$$

Here F and G are any two functions on phase space. In particular, the Wigner function $W_n(k,x)$ (to be defined in a moment) associated with mode n is invariant along its rays in the absence of mode conversion: $dW_n/d\sigma_n=\{D_n,W_n\}_8=0$. The Wigner function is related to the action density of mode n, $J_n(\mathbf{x},\mathbf{k};t)$ (Refs. 8 and 9),

$$W_n(k,x)=J_n(\mathbf{x},\mathbf{k};t)\,\delta(D_n(k,x))\,, \tag{4}$$

where δ is the one-dimensional Dirac delta function. Equation (4) can be taken as the *definition* of the action density. Inserting this expression into the Poisson bracket we find

$$[(\partial_\omega D_n)\partial J_n/\partial t+\{D_n,J_n\}_6]\delta(D_n)=0\,.$$

The new Poisson bracket, $\{,\}_6$, is restricted to the six-dimensional subspace $(\mathbf{k},\mathbf{x})$. On the manifold $D_n=0$ we have

$$\begin{aligned}\partial D_n/\partial\mathbf{x}&=-(\partial_\omega D_n)(\partial\Omega_n/\partial\mathbf{x})\,,\\ \partial D_n/\partial t&=-(\partial_\omega D_n)(\partial\Omega_n/\partial t)\,,\\ \partial D_n/\partial\mathbf{k}&=-(\partial_\omega D_n)(\partial\Omega_n/\partial\mathbf{k})\,.\end{aligned}\tag{5}$$

Using these results we obtain $[\partial J_n/\partial t+\{J_n,\Omega_n\}_6]\delta(D_n)=0$, where Ω_n is the dispersion relation [Eq. (3)] for mode n. On the surface $D_n=0$ this implies that J_n is a ray invariant:

$$\frac{dJ_n}{dt_n}\equiv\frac{\partial J_n}{\partial t}+\{J_n,\Omega_n\}_6=0\,.\tag{6}$$

We now modify the wave-kinetic equations for modes a and b to allow for mode conversion. Our starting point is Eq. (2). The rays of mode a lie on the seven-dimensional surface $D_a(k,x)=0$. As long as D_b is not zero along ray a's orbit we can ignore it. However, if ray a intersects the surface $D_b=0$ then the two modes interact and exchange energy, momentum, and action (see Fig. 1). It is important to note that this conversion process from mode a to mode b leads to a definite outgoing ray for mode b. We solve the general mode-conversion problem, allowing both the incoming wave fields to be nonvanishing. Near the conversion point, (k_c,x_c), we expand D_n and η' in Taylor series as

$$D_n(k,x)\approx(\partial D_n/\partial x)(x-x_c)+(\partial D_n/\partial k)(k-k_c)\,,$$

and $\eta'(k,x)\approx\eta'(k_c,x_c)$. The derivatives are evaluated at (k_c,x_c). We use the notation of Ref. 5 throughout. Following Ref. 5 we make a linear canonical change of phase-space coordinates from (k,x) to (p,q) according to $p_1\equiv-B^{-1/2}D_a(k,x)$ and $q_1=B^{-1/2}D_b(k,x)$. (This definition differs slightly from that of Ref. 5 in order to simplify some of the ensuing algebra.) The constant B is determined by requiring $\{q_1,p_1\}_8=1$ which implies that $B=\{D_a,D_b\}_8$. We require $B>0$. (If $B<0$, we relabel $a\leftrightarrow b$.) Using these definitions we have

$$\frac{dq_1}{d\sigma_a}=B^{1/2},\quad\frac{dp_1}{d\sigma_a}=0;\quad\frac{dq_1}{d\sigma_b}=0,\quad\frac{dp_1}{d\sigma_b}=B^{1/2}.$$

The other canonical coordinates $(q_2,p_2,q_3,p_3,q_4,p_4)$ exist by Darboux's theorem.[11] We do not need their explicit form; but we note that they are all invariant along the rays: $dq_k/d\sigma_n=dp_k/d\sigma_n=0$ $(k=2,3,4;\ n=a,b)$. Mode a enters from $q_1=-\infty$, passes through the mode-conversion region surrounding $q_1=0$, and exits to $q_1=+\infty$. Similarly, mode b enters from $p_1=-\infty$, interacts in the neighborhood of $p_1=0$, and exits to $p_1=+\infty$. In the new canonical coordinates the dispersion matrix $\mathbf{D}$ takes the form

$$B^{1/2}\begin{pmatrix}-p_1&\eta\\ \eta^*&q_1\end{pmatrix},\tag{7}$$

where $\eta=B^{-1/2}\eta'$. Using a *metaplectic transformation*[12] (whose explicit form is not needed in this paper) and (7), we express Eq. (2) in the q representation by setting $p_1\to-i\partial/\partial q_1$,

$$\begin{pmatrix}i\partial/\partial q_1&\eta\\ \eta^*&q_1\end{pmatrix}\begin{pmatrix}Z_a(q)\\ Z_b(q)\end{pmatrix}=0\,.\tag{8}$$

We eliminate $Z_b(q)$ to get a single equation for Z_a:

$$dZ_a/Z_a=-i|\eta|^2dq_1/q_1\,.$$

Care must be taken because of the pole at $q_1=0$. At this point our analysis diverges from that given in Ref. 5 since we wish to allow *both* incoming waves to be nonzero. We integrate this for real q_1 separately in the regions $q_1<0$ and $q_1>0$:

$$Z_a(q)=\begin{cases}\alpha_-|q_1|^{-i|\eta|^2}&(q_1<0),\\ \alpha_+q_1^{-i|\eta|^2}&(q_1>0)\,.\end{cases}\tag{9}$$

The integration constants $\alpha_\pm$ are functions of q_2, q_3, and q_4 and are closely related to the incoming and outgoing field amplitudes. Using Eq. (8) we write $Z_b(q)=-\eta^*Z_a(q)/q_1$. The p_1 representations of Z_a and Z_b can be constructed from their q representations since (q_1,p_1) play the role of a Fourier-transform pair, holding (q_2,q_3,q_4) fixed. We carry out the transform explicitly only for Z_b since that for Z_a is similar:

$$Z_b(p_1)=(2\pi)^{-1/2}\int dq_1e^{-ip_1q_1}Z_b(q_1)\,.$$

The integral is broken into separate parts for $q_1<0$ and $q_1>0$. The resulting integrals can be expressed in terms

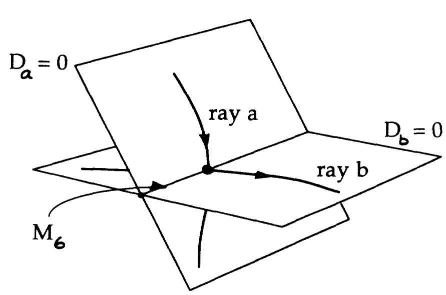

FIG. 1. A schematic diagram indicating the transverse intersection of the two dispersion manifolds in the eight-dimensional ray phase space. The mode-conversion manifold, M_6, is the six-dimensional set of points which form this intersection.

of Γ functions:

$$Z_b(p_1)=\begin{cases}-\eta^* |p_1|^{i|\eta|^2}[\alpha_+ e^{\pi|\eta|^2/2}-\alpha_- e^{-\pi|\eta|^2/2}]\Gamma(-i|\eta|^2), & p_1<0,\\ -\eta^* p_1^{i|\eta|^2}[\alpha_+ e^{-\pi|\eta|^2/2}-\alpha_- e^{\pi|\eta|^2/2}]\Gamma(-i|\eta|^2), & p_1>0.\end{cases} \tag{10}$$

We now relate the incoming and outgoing *action amplitudes*, which are defined as

$$\bar{Z}_a(\pm q_1)\equiv |q_1|^{i|\eta|^2}Z_a(\pm q_1),$$
$$\bar{Z}_b(\pm p_1)\equiv |p_1|^{-i|\eta|^2}Z_b(\pm p_1).$$

Equations (9) and (10) allow us to construct the *S matrix*:

$$\begin{bmatrix}\bar{Z}_a(+q_1)\\ \bar{Z}_b(+p_1)\end{bmatrix}=\begin{bmatrix}\tau & -\beta\\ \beta^* & \tau\end{bmatrix}\begin{bmatrix}\bar{Z}_a(-q_1)\\ \bar{Z}_b(-p_1)\end{bmatrix}, \tag{11}$$

i.e., $\bar{\mathbf{Z}}_{\rm out}=\mathbf{S}\cdot\bar{\mathbf{Z}}_{\rm in}$. Here $\tau(\eta)\equiv\exp(-\pi|\eta|^2)$ and $\beta(\eta)\equiv(2\pi\tau)^{1/2}/\eta\Gamma(-i|\eta|^2)$. Straightforward algebra shows that the S matrix is unitary. This requires identity 6.1.29 in Ref. 13, $|\Gamma(i|\eta|^2)|^2=\pi/|\eta|^2\sinh(\pi|\eta|^2)$.

From the wave-kinetic viewpoint, conversion takes place when rays a and b cross the singular surface M_6. Thus we use the above results to relate the incoming and outgoing action densities, J_n, for the two modes. This is done by using Eqs. (9) and (10) to relate the incoming and outgoing Wigner functions, W_n, and then making the identification (4).

The Wigner function associated with $Z_n(q)$ is defined as

$$W_n(p,q)\equiv\int d^4s\, e^{-ps}Z_n(q+\tfrac{1}{2}s)Z_n^*(q-\tfrac{1}{2}s).$$

Because the Wigner function is invariant under canonical transformations[12] it can be computed in whatever representation simplifies the analysis. After some lengthy analysis (given in a later paper) it is possible to show that the incoming and outgoing action densities obey relations of the form

$$J_a^{\rm out}=TJ_a^{\rm in}+(1-T)J_b^{\rm in}+\Lambda,$$
$$J_b^{\rm out}=TJ_b^{\rm in}+(1-T)J_a^{\rm in}-\Lambda,$$

where the transmission coefficient, $T\equiv\tau^2=\exp(-2\pi\times|\eta|^2)$, is the same in both equations. The quantity Λ depends on the incoming amplitudes and includes *phase interference* effects. We now specialize the discussion to *incoherent* incoming wave fields. An important physical example is given by waves whose sources are turbulent processes or thermal emission. We can model such situations by decomposing the incoming wave fields into a gas of wave packets, where each packet has a phase chosen at random in the interval $[0,2\pi)$. Performing an ensemble average, the phase interference terms vanish and we arrive at the jump conditions for the ensemble-averaged action densities:

$$\langle\langle J_a^{\rm out}\rangle\rangle=T\langle\langle J_a^{\rm in}\rangle\rangle+(1-T)\langle\langle J_b^{\rm in}\rangle\rangle,$$
$$\langle\langle J_b^{\rm out}\rangle\rangle=T\langle\langle J_b^{\rm in}\rangle\rangle+(1-T)\langle\langle J_a^{\rm in}\rangle\rangle. \tag{12}$$

We now modify the wave-kinetic equations for modes a and b to allow for a discontinuous change across the mode-conversion manifold. In what follows we drop the $\langle\langle\rangle\rangle$ for notational convenience. All the following relations apply to the *ensemble-averaged* action densities. We modify Eq. (6) for modes a and b to include the coupling due to mode conversion:

$$\frac{dJ_a(\mathbf{x},\mathbf{k};t)}{dt_a}=\gamma[J_b^{\rm in}(\mathbf{x},\mathbf{k};t)-J_a^{\rm in}(\mathbf{x},\mathbf{k};t)]\times\delta(\Omega_b(\mathbf{x},\mathbf{k};t)-\Omega_a(\mathbf{x},\mathbf{k};t)), \tag{13a}$$

$$\frac{dJ_b(\mathbf{x},\mathbf{k};t)}{dt_b}=-\gamma[J_b^{\rm in}(\mathbf{x},\mathbf{k};t)-J_a^{\rm in}(\mathbf{x},\mathbf{k};t)]\times\delta(\Omega_b(\mathbf{x},\mathbf{k};t)-\Omega_a(\mathbf{x},\mathbf{k};t)). \tag{13b}$$

The coupling coefficient is $\gamma=B(1-T)(\partial_\omega D_a\partial_\omega D_b)^{-1}$. $J_n^{\rm in}(\mathbf{x},\mathbf{k};t)$ is the value of J_n as it enters the mode-conversion region:

$$J_n^{\rm in}(\mathbf{x},\mathbf{k};t_c)\equiv\lim_{t\uparrow t_c}J_n(\mathbf{x}(t),\mathbf{k}(t);t).$$

Equations (13) satisfy the desired jump conditions across the mode-conversion surface (M_6). This is shown by integrating them across the singular region. For example, with mode a (we have assumed the jump occurs at $t_a=0$ for simplicity),

$$\begin{aligned}J_a^{\rm out}-J_a^{\rm in}&=\lim_{\epsilon\to0}\left(\int_{-\epsilon}^{\epsilon}\frac{dJ_a}{dt_a}dt_a\right)\\ &=\gamma(J_b^{\rm in}-J_a^{\rm in})\int_{-\epsilon}^{\epsilon}\delta(\Omega_b-\Omega_a)dt_a\\ &=\gamma(J_b^{\rm in}-J_a^{\rm in})\left|\frac{d(\Omega_b-\Omega_a)}{dt_a}\right|^{-1}_{t_a=0}.\end{aligned}$$

Use of

$$d(\Omega_b-\Omega_a)/dt_a\equiv\partial(\Omega_b-\Omega_a)/\partial t+\{\Omega_b-\Omega_a,\Omega_a\}_6,$$

Eqs. (5), and $B\equiv\{D_a,D_b\}_8$ recovers Eqs. (12) after a slight rearrangement.

Notice in Eqs. (13) that action is exchanged only between waves with the same $\mathbf{k}$ and ω. This ensures momentum and energy conservation. It is also easy to prove that the total action is conserved. Consider the total action contained in the phase-space volume V: $J_{\rm tot}=\int_V d^6z(J_a+J_b)$. The phase-space volume element is defined as $d^6z \equiv d^3x\, d^3k/(2\pi)^3$. By taking the time derivative of this expression, using Eqs. (13), it is straightforward to show that $J_{\rm tot}$ changes with time only due to action passing into or out of the volume through its surface, ∂V. Thus no action is generated or lost due to the conversion process.

Finally, we wish to note that incoherent mode conversion provides a mechanism for the production of wave entropy. Since the wave-kinetic formulation of eikonal wave propagation is reminiscent of particle kinetics it is interesting to note that the assumption of uncorrelated incoming phases plays a role analogous to Boltzmann's assumption of molecular chaos in particle kinetics. This assumption leads to an H theorem and macroscopic irreversibility, although the microscopic equations are reversible.

The wave entropy is related to the action density as follows:[14]

$$S(J_a,J_b) \equiv \int_{V_a} d^6z \ln J_a + \int_{V_b} d^6z \ln J_b \,.$$

These integrals are restricted to regions of phase space where J_a and J_b are nonzero. This restriction arises naturally from the derivation of the wave entropy given in Ref. 14. Here we simply take the restriction as required to make S well defined.

Taking the time derivative of S, a little algebra leads to

$$\frac{dS}{dt} = \int \frac{d^3k}{(2\pi)^3} \int d^3x \frac{|B|(1-T)}{|\partial_\omega D_a|\,|\partial_\omega D_b|} \frac{(J_a^{\rm in} - J_b^{\rm in})^2}{J_a^{\rm in} J_b^{\rm in}} \times \delta(\Omega_b - \Omega_a)\,.$$

For fixed $\mathbf{k}$ and t the condition $\Omega_a = \Omega_b$ defines a two-dimensional surface in space (call it M_s). Choose local coordinates on this surface (s_1,s_2,s_3) such that (s_1,s_2) lie in M_s and s_3 is in the perpendicular direction. Carrying out the s_3 integral leads to

$$\frac{dS}{dt} = \int \frac{d^3k}{(2\pi)^3} \int_{M_s} d^2s \frac{|B|(1-T)}{|\partial_\omega D_a|\,|\partial_\omega D_b|} \frac{(J_a^{\rm in} - J_b^{\rm in})^2}{J_a^{\rm in} J_b^{\rm in}} \times \left| \frac{\partial(\Omega_b - \Omega_a)}{\partial s_3} \right|^{-1} .$$

Notice that the time derivative of the entropy is positive unless $J_a^{\rm in} = J_b^{\rm in}$ everywhere on M_s.

In summary, we have shown how recent developments in the theory of linear mode conversion can be used to include conversion in the wave-kinetic formalism. Both wave modes obey a wave-kinetic equation away from the mode-conversion surface and undergo a discontinuous jump across that surface. The jump conditions are determined by using the results of a general mode-conversion calculation assuming that the incoming waves are incoherent. This leads to the production of wave entropy in the mode-conversion region.

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