Self-Organized Critical State of Sandpile Automaton Models

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(Received 21 November 1989)

We study a general Bak-Tang-Wiesenfeld–type automaton model of self-organized criticality in which the toppling conditions depend on local height, but not on its gradient. We characterize the critical state, and determine its entropy for an arbitrary finite lattice in any dimension. The two-point correlation function is shown to satisfy a linear equation. The spectrum of relaxation times describing the approach to the critical state is also determined exactly.

PACS numbers: 05.40.+j, 05.60.+w, 46.10.+z, 64.60.−i

Recently Bak, Tang, and Wiesenfeld (BTW) have introduced a simple automaton model that captures some important features of the dynamics of sandpiles.[1] It displays the remarkable property that starting from an arbitrary initial state, its stochastic evolution produces at long times a unique critical state characterized by power-law correlations in space and time. This state has been called the self-organized critical (SOC) state, and BTW have argued that such models provide a natural framework to describe diverse phenomena involving dissipative, nonlinear transport in open systems, such as $1/f$ noise in electrical networks, distribution of visible matter in the Universe, earthquakes, etc.[2]

To determine the critical exponents characterizing the SOC state, BTW have relied mainly on numerical simulations,[1] and the mean-field approximation.[3] More extensive simulations have been made by Kadanoff *et al.*[4] and Manna and Grassberger.[5] A directed version of this problem has been solved exactly by Dhar and Ramaswamy,[6] and its critical exponents are known in all dimensions. Using some assumptions about compactness of avalanche clusters, Zhang[7] has determined the critical exponents of the undirected BTW model in all dimensions. For the cluster-size exponent τ (probability that adding a particle at random causes exactly n topplings $\sim n^{-\tau}$), he finds that $\tau = 2(1-1/d)$ for $1 \le d < \infty$. For $d=2$, this value disagrees with the numerical estimate $\tau \approx 1.22$ obtained in Ref. 5. Obukhov[8] has argued that the avalanche process may be viewed as a branching self-avoiding walk, and using ϵ-expansion techniques he finds that the upper critical dimension for the undirected BTW model is 4, and $\tau = \frac{3}{2}$ for $d \ge 4$. Hwa and Kardar[9] have studied the evolution of a nonlinear continuum model in the presence of noise, which they argue is in the same universality class as the BTW model with a preferred direction, but the upper critical dimension (and also the critical exponents) in this model differs from its value in the model studied in Ref. 6.

In this Letter, we study the SOC state of a general BTW-type automaton model on an arbitrary finite set of sites. We consider the case when the toppling at a site occurs if the "height" at the site exceeds some locally prescribed critical value. Our treatment is valid even when the toppling rules differ from site to site, and for any dimension of the lattice. In the following, we shall refer to this model as the Abelian model (AM) to distinguish it from other BTW-type models in which the operators do not satisfy a commutative algebra when the toppling criteria depend on gradients of height. We show that in the SOC state of the AM, some stable configurations are forbidden, but all the *allowed* configurations occur with equal probability. The number of allowed configurations is shown to be equal to the determinant of an integer matrix Δ that specifies the evolution rules. The two-point correlation functions of the model are given by the matrix Δ^{-1}. We define operators corresponding to adding a particle at different sites and show that they commute with each other. The algebra of these operators is used to determine the spectrum of relaxation times of the AM.

Definition of the model.— We consider a set of N sites, labeled by integers 1 to N. To each site i is assigned an integer variable z_i. The AM is specified by two rules.

(i) Adding a particle: We select a site at random (probability of selecting site i being p_i, all p_i's not necessarily equal) and increase z_i by 1. Other z_j's ($j \ne i$) are unchanged.

(ii) The toppling rule: This is specified in terms of an $N \times N$ integer matrix Δ, and a set of N threshold values z_{ic} ($i=1$ to N). If any $z_i > z_{ic}$, that site *topples*, and in the language of sandpiles, some of the particles from the toppled site drop onto other sites, and some may leave the system. On toppling at site i,

$$z_j \to z_j - \Delta_{ij}, \text{ for } j = 1 \text{ to } N. \tag{1}$$

The integer matrix Δ satisfies the conditions

$$\Delta_{ii} > 0, \text{ for all } i, \tag{2}$$

$$\Delta_{ij} \le 0, \text{ for all } i \ne j, \tag{3}$$

and

$$\sum_{j=1}^{N} \Delta_{ij} \ge 0, \text{ for all } i. \tag{4}$$

These conditions ensure that on toppling at site i, z_i must decrease, z_j for $j \neq i$ cannot decrease, and there is no creation of particles in the toppling process. Particles *can* leave the system, say at the edges; in fact, no steady state is possible otherwise. We assume in addition that the matrix Δ is such that any configuration relaxes to a stable configuration in a finite number of steps. We do *not* assume that the matrix Δ is symmetrical. The conventional undirected nearest-neighbor BTW model[1,5] as well as its directed and partially directed variants[9] are obtainable as special cases of the AM.

Without loss of generality, we can assume that $z_{ic} = \Delta_{ii}$ for all i. Then, any configuration $\{z_i\}$ in which $1 \leq z_i \leq \Delta_{ii}$ is a stable configuration under the toppling rule. We define N operators a_i ($i=1$ to N) on this space of stable configurations by requiring that $a_i C$ be the stable configuration obtained by adding a particle at site i to the configuration C and allowing the system to evolve by toppling.

Consider an unstable configuration in which two sites α and β are both critical (i.e., $z_\alpha > \Delta_{\alpha\alpha}$ and $z_\beta > \Delta_{\beta\beta}$). Then first toppling α leaves β critical [Eq. (3)], and after toppling both α and β we get a configuration in which z_i decreases by $\Delta_{\alpha i} + \Delta_{\beta i}$ for $i=1$ to N. This is clearly symmetrical under exchange of α and β. Thus we get the same resulting configuration irrespective of whether α or β is toppled first. By a repeated use of this argument, we see that in an avalanche, the same final stable configuration is reached irrespective of the sequence in which unstable sites are toppled. Also toppling at an unstable site α, and then adding a particle at site β gives the same result as first adding at β, and then toppling at α. From these two properties it follows that for all configurations C, and all i and j, we get $a_i a_j C = a_j a_i C$. In other words the operators a_i commute with each other:

$$[a_i, a_j] = 0, \quad \text{for all } i,j . \tag{5}$$

Note that in BTW models with toppling condition depending on gradients, the inequality (3) is not satisfied and the operators a_i do not commute.

The commutativity of a's is the crucial property underlying the tractability of AM. In particular, it implies that the steady state (the SOC state) of the model can be characterized very simply.[10] From the general theory of Markov chains, the set of all stable configurations can be divided into two classes: recurrent and transient. We define a configuration C to be recurrent, iff there exist positive integers m_i ($i=1$ to N) such that

$$a_i^{m_i} C = C, \quad \text{for all } i . \tag{6}$$

From Eq. (5), it is easy to see that if C is recurrent then configurations $a_i C$ ($i=1$ to N) are also recurrent.[11] We denote the set of all recurrent configurations by $\mathcal{R}$. It follows that $\mathcal{R}$ is closed under multiplication by operators a_i. Once our system gets into a recurrent configuration, it can never get out of $\mathcal{R}$ under the Markovian evolution of AM. It follows that all nonrecurrent configurations are *transients*, and have zero probability of occurrence in the SOC state.

For operators a_i restricted to domain $\mathcal{R}$, inverses can be defined. For any configuration C satisfying Eq. (6), we define

$$a_i^{-1} C = a_i^{m_i - 1} C, \quad \text{for all } i = 1 \text{ to } N . \tag{7}$$

As argued in Ref. 9, the existence of a unique inverse in the set of recurrent configurations implies that the state in which all recurrent configurations occur with equal probability is the invariant state of the Markovian evolution. We thus have a full characterization of the SOC state of AM: In it only recurrent configurations have a nonzero probability of occurrence, and this nonzero value is the same for all recurrent configurations.

Consider now any configuration $C \in \mathcal{R}$, to which we add Δ_{ii} particles at some site i one after another. Since $z_i > 0$ in C, after these additions the site will become unstable, and topple, in the process adding $(-\Delta_{ij})$ particles at all other sites j ($j \neq i$). We thus see that the operators a_i ($i=1$ to N) satisfy the equations

$$a_i^{\Delta_{ii}} = \prod{}' a_j^{-\Delta_{ij}} , \tag{8}$$

where the primed product sign indicates product over all $j \neq i$. Equivalently, we write

$$\prod_{j=1}^{n} a_j^{\Delta_{ij}} = 1, \quad \text{for all } i = 1 \text{ to } N . \tag{9}$$

Since the a's commute with each other, all representations of the algebra given by Eq. (9) are one-dimensional. We write

$$a_j = \exp(i\phi_j), \quad j = 1 \text{ to } N , \tag{10}$$

where ϕ_j are some real numbers. In terms of ϕ's, Eq. (9) can be written as

$$\sum_{j=1}^{N} \Delta_{ij} \phi_j = 2\pi n_i, \quad \text{for all } i , \tag{11}$$

where n_i, $i=1$ to N, are some integers. Solving Eq. (11) we get

$$\phi_i = 2\pi \sum_{j=1}^{N} [\Delta^{-1}]_{ij} n_j, \quad \text{for all } i . \tag{12}$$

Equation (12) shows that the allowed values of $\{\phi_i\}$ form a periodic lattice in the N-dimensional space. For each choice of $\{n_i\}$, we have a set of values $\{\phi_i\}$, which gives a representation of the operator algebra of Eq. (9). But ϕ_i's are phases, and are defined only modulo 2π. Thus in Eq. (12), only points lying with the N-dimensional cube $0 \leq \phi_i < 2\pi$ ($i=1$ to N) give rise to distinct representations. The number of such representations is the ratio of volumes of the cube and the volume of the unit cell of the ϕ lattice. This number is also equal to the number of distinct elements of the algebra, i.e., products of the type $a_1^{m_1} a_2^{m_2} \cdots a_N^{m_N}$ ($m_1, m_2, \ldots$ are non-negative integers) which are not equal under Eq. (9). But each such element acting on a configuration C gives rise to a distinct configuration. Hence, this number must equal $\mathcal{N}_R$, the number of distinct configurations in $\mathcal{R}$. Thus we get

$$\mathcal{N}_R = \mathrm{Det}\Delta . \tag{13}$$

Since all these configurations occur with equal probability in the SOC state, the entropy of the SOC state S is given by

$$S = \ln \mathrm{Det}\Delta . \tag{14}$$

For the directed models, the matrix Δ is upper triangular, and one recovers the result of Ref. 9,

$$\mathcal{N}_R = \prod_{i=1}^{N} \Delta_{ii} . \tag{15}$$

As another example, for the undirected nearest-neighbor AM on the square lattice with free boundary conditions studied in Refs. 1 and 5, we get for s, the entropy per site of the SOC state in the thermodynamic limit,

$$s = (2\pi)^{-2} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \ln(4 - 2\cos\theta - 2\cos\phi) . \tag{16}$$

Equation (13) can be obtained more directly. In the space of all possible (including unstable) configurations obtainable from $\mathcal{R}$ by addition of particles, we define two configurations $\{z_i\}$ and $\{z_i'\}$ as equivalent iff under topplings, they evolve to the same stable configuration. Then, if $\{z_i\}$ and $\{z_i'\}$ are equivalent, there exist some integers r_i ($i = 1$ to N) such that

$$z_j' = z_j - \sum_{j=1}^{N} r_i \Delta_{ij}, \text{ for all } j . \tag{17}$$

Thus if we represent $\{z_i\}$'s by points of a N-dimensional hypercubical lattice with basis vectors $\bar{e}_i$, the set of equivalent points forms a superlattice with basis vectors $\sum_{j=1}^{n} \Delta_{ij} \bar{e}_j$ ($i = 1$ to N). Since to each equivalence class of configurations there corresponds a unique recurrent configuration,[12] the volume of the unit cell of the superlattice must equal $\mathcal{N}_R$. This again gives Eq. (13).

It is desirable to get a more direct characterization of the set $\mathcal{R}$. We define a forbidden subconfiguration (FSC) as any set F of r sites ($r \geq 1$) if the corresponding height variables $\{z_{\alpha_j}\}$, $j \in F$, satisfy the inequalities

$$z_j \leq \sum_i{}' (-\Delta_{ij}), \text{ for all } j \in F , \tag{18}$$

where the primed summation denotes sum over all $i \in F$ with $i \neq j$. A (stable or unstable) configuration that contains no FSC's is called an allowed configuration. We argue below that the set of stable allowed configurations is the same as $\mathcal{R}$. A more explicit characterization of $\mathcal{R}$ is needed for calculation of various ensemble-averaged quantities in the SOC state such as the average height, probability distributions of avalanches by their mass, durations, etc. This has been accomplished so far only for the SOC state on a Bethe lattice, the details of which will be reported in a future publication.[13] We find that the probability that avalanche contains more than n sites varies as $n^{-1/2}$ for n. The probability that its duration exceeds t varies as t^{-1} for large t.

There is a simple recursive procedure to determine if a given configuration is allowed. We consider a test set T of sites. In the beginning, T consists of all the sites of the lattice. We test the hypothesis $F = T$, using the inequalities (18). If these are satisfied for all sites in T, then the hypothesis is true and the configuration disallowed. Otherwise, there are some sites for which the inequality is violated. These sites cannot be part of any FSC, as the inequalities (18) will remain unsatisfied if T is replaced by any smaller subset in its right-hand side. Deleting these sites from T, we get a smaller test set T', and repeat the previous procedure. In the end, we either get a finite FSC F, else the set T becomes empty, and then the configuration is allowed.

We now show that the set of allowed, stable configurations is closed under the dynamics of AM. Assume the contrary. Then there exists an allowed configuration C such that by a single toppling it becomes the configuration C' which contains a FSC F (adding particles only increases heights, and cannot create a FSC). If this toppling occurs at site i, from the toppling rule (1) one gets that if F is a FSC in C', then the set obtained by deleting i from F is a FSC in C. This contradicts our assumption that C is allowed. Hence the set of allowed configurations is closed. The recurrent configuration with $z_i = \Delta_{ii}$ for all i, is clearly allowed [by Eq. (4)]. Since all recurrent configurations are reachable from this particular one, it follows that all recurrent configurations are allowed. The converse statement appears quite plausible, though a strict proof is lacking. Together, these would imply that the set of allowed stable configurations is $\mathcal{R}$.

There is an interesting relationship between FSC's and avalanche clusters. To see this, consider a recurrent configuration C, and let us try to construct the $C' = a_i^{-1} C$. Let C'' be the configuration obtained by decreasing z_i in C by 1, and leaving other sites unchanged. If C'' is allowed, then $C' = C''$, by uniqueness of inverse. If C'' is disallowed, it must contain an FSC, say F such that $i \in F$. Since the inequality (18) at site j is preserved by toppling at all other sites, it follows that F is a subset of the cluster of sites in C' where at least one toppling occurs on adding a particle at i. In fact, F is the set of sites where as many topplings occur as at i.

We now calculate the two-point correlation function in the SOC state. Let G_{ij} be the expected number of topplings at site j, due to the avalanche caused by adding a particle at i. Then the total average flux of particles out of site j is $G_{ij} \Delta_{jj}$. The total average flux of particles into site j is $\sum_k G_{ik} (-\Delta_{kj})$, where k is summed over all sites

$\neq j$. In the SOC state, the average influx must equal the average outflux, and we get

$$\sum_k G_{ik}\Delta_{kj}=\delta_{ij} \tag{19}$$

and

$$G_{ij}=[\Delta^{-1}]_{ij}, \text{ for all } i,j. \tag{20}$$

An analog of Eq. (19) for the coarse-grained energy density was already written down by inspection in Ref. 7, when the matrix Δ is a Laplacian operator. Note that the argument leading to Eq. (19) is quite general. The equation would hold even in a non-Abelian case when the toppling condition depends on gradients.

As a simple application of Eq. (20), let us calculate $\langle T\rangle$, the average number of topplings per added particle on a finite $L\times L$ lattice for the square-lattice nearest-neighbor AM. If a toppling occurs at the edge (corner), one (two) of the particles leaves the system. Writing down the matrix Δ, a straightforward calculation gives

$$\langle T\rangle=\frac{1}{L^2(L+1)^2}\sum_{m,n}\cot^2\frac{\pi m}{2(L+1)}\cot^2\frac{\pi n}{2(L+1)}\left[\sin^2\frac{m\pi}{2(L+1)}+\sin^2\frac{n\pi}{2(L+1)}\right]^{-1}, \tag{21}$$

where the summation over m,n extends over all *odd* integers $1\leq m\leq L$, $1\leq n\leq L$. For large L, we get $\langle T\rangle\sim L^2$. This formula agrees with the recent numerical estimates of $\langle T\rangle$ within numerical uncertainties.[5]

Finally, we determine the spectrum of relaxation to the SOC state. Let $P_t(C)$ be the probability that the stable configuration obtained after the tth particle has been added is C. Then these satisfy the master equation

$$P_{t+1}(C)=\sum_{C'}W(C,C')P_t(C'). \tag{22}$$

The rates $W(C,C')$ can be written in terms of an $\mathcal{N}_R$-dimensional matrix W, which in terms of operators $\{a_i\}$ is given by

$$W=\sum_{i=1}^{N}p_i a_i. \tag{23}$$

Since the operators $\{a_i\}$ are simultaneously diagonalizable with eigenvalues given by Eqs. (10) and (12), the eigenvalues of W are completely determined.

The choice $n_i=0$ for all i in Eq. (12) corresponds to the steady state with eigenvalue 1. The next largest eigenvalue determines the relaxation time of the slowest decaying fluctuations in the SOC state. For the undirected d-dimensional AM with nearest-neighbor drops only, we find that this relaxation time varies as L^d, where L is the linear extent of the system. Note that, for large L, this is much larger than the average duration of an avalanche τ which satisfies $\tau\leq\langle T\rangle\sim L^2$. In fact, in the numerical simulations of AM, one can speed up relaxation in the SOC state substantially by using the commutativity of a's, and evolve by toppling only after (say) every Nth particle is added.

I thank R. Ramaswamy for many discussions and correspondence, and M. Barma and T. R. Ramadas for useful comments on the manuscript.

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[10]We use the word "state" to denote an ensemble of configurations with specified probabilities.

[11]Thus the integers m_i are independent of C, and using their product, can be chosen to be independent of i as well.

[12]The geometrical shape of the set $\mathcal{R}$ in this representation is quite nontrivial, but copies of $\mathcal{R}$ can be arranged to give a simple periodic tiling of the N-dimensional space.

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