Quantum Geometrodynamics of the Open Topological Membrane and String Moduli Space

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We show that by coupling a three-dimensional Chern-Simons theory to (2+1)-dimensional gravity through an arbitrarily small F^2 term, one can obtain integrals of conformal-field-theory amplitudes over moduli space. The conformal anomaly appears as an induced gravitational Chern-Simons term. String-theory amplitudes can thus be obtained from three dimensions.

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(1) Introduction.— In the past year, it has become increasingly apparent that three-dimensional topological theories can provide valuable insights into the structure of two-dimensional conformal field theories. In particular, as Witten and others have shown,[1,2] the partition functions and conformal blocks of rational conformal field theories can be obtained from three-dimensional Chern-Simons theories. Such topological theories may, in turn, be viewed as the infrared limits of dynamical three-dimensional theories, that is, theories of three-dimensional membranes. One has a choice of philosophies: one may start with a Chern-Simons theory and introduce a higher-derivative term of the form $(1/\gamma)F^2$ to regulate amplitudes, or one may start with a full dynamic theory and view the topological sector as arising in a suitable low-energy limit.

So far, work in this field has focused on the derivation of conformal blocks for Wess-Zumino-Novikov-Witten (WZNW) models on surfaces with fixed complex structures. But for many physicists, the real interest in conformal field theory comes from its connection to string theory, in which one integrates over the moduli space of complex structures. The aim of this paper is to show that such an integral over moduli is a natural outcome of coupling a three-dimensional Chern-Simons theory to (2+1)-dimensional gravity. Since so many two-dimensional conformal field theories—including all known rational conformal field theories[2]—arise from Chern-Simons theory, we regard this as a step toward a three-dimensional topological picture of strings.

(2) Complex structure in Chern-Simons theory.— Let M be a three-manifold with the topology $[0,1]\times\Sigma$, where Σ is a Riemann surface. The boundary $\partial([0,1]\times\Sigma)$ consists of two copies of Σ; as observed by Kogan[3] and Elitzur *et al.*,[4] the corresponding WZNW model includes left movers coming from one boundary component and right movers coming from the other. Vertex operators are naturally incorporated by including Wilson lines which begin on one boundary and end on the other.

We begin with a paradox. The Chern-Simons action

$$S_1=\frac{k}{4\pi}\int_M (A\wedge dA+\tfrac{2}{3}A\wedge A\wedge A) \tag{2.1}$$

is topological, independent of any metric. Yet Chern-Simons theory on M gives rise to a WZNW model on ∂M, which requires a choice of complex structure for its description. Wave functions in the Chern-Simons theory take the form $\Psi[A_{\bar z}]$, where $A_{\bar z}$ is a boundary component of the Chern-Simons gauge field. But such a form clearly requires a metric on ∂M, or at least a conformal equivalence class of metrics, to define $\bar z$.

Of course, this does not mean that Chern-Simons theory is inconsistent. The Chern-Simons action is first order in time derivatives, and to canonically quantize one must choose a polarization, i.e., a specification of which components of A are position variables and which are momenta. Such a choice introduces a complex structure. Equivalently, from the point of view of the path integral, one must select appropriate boundary conditions for A: this again requires picking out one component whose boundary value is given on ∂M. It can be shown that the Hilbert spaces arising from different choices of polarization are unitarily equivalent, and that expectation values of Wilson lines in closed three-manifolds are independent of any such choices.[4,5] If our aim is to derive string-theory amplitudes from the Chern-Simons theory, however, we must introduce an integration (of nonchiral amplitudes) over complex structures; that is, we must promote the dependence on complex structure to something of physically significance.

To do this, let us observe first that perturbative Chern-Simons theory requires a gauge-invariant regularization of the Chern-Simons action to define off-shell amplitudes. One way to regularize (2.1) is to add a higher-derivative term[6-8]

$$S_2=-\frac{1}{4\gamma}\int_M F\wedge *F = -\frac{1}{4\gamma}\int_M \sqrt{-g}\,g^{ac}g^{bd}F_{ab}F_{cd}\,, \tag{2.2}$$

where we eventually take $\gamma\rightarrow\infty$ to recover the pure Chern-Simons action. We may alternatively view (2.2) as a genuine contribution to the three-dimensional action for a topologically massive gauge theory,[3,9] the fundamental theory is then one of a three-dimensional membrane, which has Chern-Simons theory as an effective low-energy limit.

The action (2.2) depends explicitly on the three-metric g_{ab}, which in the Arnowitt-Deser-Misner formalism is given by

$$ds^2 = -(N\,dt)^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,, \tag{2.3}$$

where h_{ij} is a metric on Σ. We should therefore expect quantum corrections to induce a (2+1)-dimensional gravitational action,

$$S_3 = \frac{1}{\kappa^2}\int_M R\sqrt{-g}\,. \tag{2.4}$$

In general, a gravitational Chern-Simons term[9]

$$S_4 = \frac{k'}{8\pi}\epsilon_{abc}(R_{ab\alpha\beta}\omega_c{}^{\alpha\beta} + \tfrac{2}{3}\,\omega_{a\alpha}{}^{\beta}\omega_{b\beta}{}^{\gamma}\omega_{c\gamma}{}^{\alpha}) \tag{2.5}$$

will also be induced. Here $\omega_{a\alpha}{}^{\beta}$ is the spin connection for the metric g_{ab}, and (2.5) can be interpreted as a Chern-Simons term for an SO(2,1) gauge theory with connection ω. The full action is thus $S_1+S_2+S_3+S_4$, plus higher-order terms in the curvature which should not affect the topological ($\gamma \to \infty$) limit.

The three-metric g_{ab} induces a two-metric, and thus a complex structure, on Σ, and we might expect this to affect our choice of polarization. Indeed, the action S_1+S_2 is now second order in time derivatives, and we can no longer view A_z and $A_{\bar z}$ as canonically conjugate. Instead, standard canonical quantization of the gauge field on $[0,1]\times\Sigma$ leads to a Hilbert space of functions $\Psi(A_z, A_{\bar z})$, with a Hamiltonian

$$\begin{aligned} H &= \frac{\gamma}{2}\frac{N}{\sqrt{h}}\left[h_{ij}\left(\pi^i - \frac{k}{4\pi}\epsilon^{ik}A_k\right)\left(\pi^j - \frac{k}{4\pi}\epsilon^{jl}A_l\right) + \frac{1}{4\gamma^2}\tilde F^2\right] \\ &= \gamma N\left[-\left(\frac{\delta}{\delta A_{\bar z}} - \frac{k}{4\pi}A_z\right)\left(\frac{\delta}{\delta A_z} + \frac{k}{4\pi}A_{\bar z}\right) + \frac{1}{4\gamma^2}\tilde F^2\right], \end{aligned} \tag{2.6}$$

where $\tilde F = \epsilon^{ij}F_{ij}$ and the complex structure in (2.6) is determined by h_{ij}. In the $\gamma \to \infty$ limit, the dominant contribution comes from the lowest-energy eigenstates, $E=0$, for which

$$\Psi[A] = \exp\left(-\frac{k}{4\pi}\int A_z A_{\bar z}\right)\Phi(A_{\bar z})\,. \tag{2.7}$$

Equation (2.7) is the proper form for a Chern-Simons wave function, but now with a particular complex structure determined by g_{ab}. The exponent determines the correct inner product, while the dependence of Φ on $A_{\bar z}$ corresponds to the polarization[4] determined by the complex structure z. This residual metric dependence of the wave functions has been studied by Wen,[10] who first pointed out that the low-energy effective action is conformally invariant, depending only on the complex structure determined by h_{ij}. The view of Chern-Simons theory as the infrared limit of a topologically massive gauge theory allows a simple interpretation of a number of features. For instance, the Hamiltonain (2.6) on a torus is equivalent to that of a particle in a constant magnetic field; the number of Chern-Simons states is fixed by the number of states in the first Landau level, and the quantization of k comes from requiring that this number be integral.[11]

(3) Three-dimensional geometrodynamics.— We have now obtained a complex structure for a WZNW model by coupling Chern-Simons theory (through the F^2 term) to (2+1)-dimensonal gravity. Can the gravitational path integral lead further to an integral over complex structures? To answer this question, observe first that the dynamics of gravity is determined completely by the constraints: A metric which satisfies the constraint equations at all times automatically satisfies the full field equations.[12] Since the constraints occur as δ functionals in the path integral, this means that the space over which we integrate is precisely the space of classical solutions with appropriate boundary values. We must therefore ask whether there is a suitable choice of boundary conditions for which the space of classical solutions gives the moduli space of complex structures on Σ.

The constraints in 2+1 dimensions can be written as[13,14]

$$\begin{aligned} \mathcal{H} &= \frac{1}{\sqrt{h}}(\pi^{ij}\pi_{ij} - \pi^2) - \sqrt{h}\,{}^{(2)}R = 0\,, \\ \mathcal{H}^i &= -2\pi^{ij}{}_{|j} = 0\,, \end{aligned} \tag{3.1}$$

where $\pi^{ij} = \sqrt{h}(K^{ij} - h^{ij}K)$ is the momentum conjugate to h_{ij} (K is the extrinsic curvature of Σ). Witten[15] has described one set of classical gravitational solutions which are in one-to-one correspondence with the Teichmüller space of Σ. These are obtained by forming the quotient of the forward light cone in $\mathbb{R}^3$ by a Fuchsian ground $\Gamma \subset SL(2,\mathbb{R})$. Equivalently, any particular solution of this type can be specified by giving an initial metric h_{ij} on Σ, and then imposing the condition

$$\pi_{ij} = \alpha\sqrt{h}\,h_{ij} \tag{3.2}$$

or, equivalently, fixing the transverse traceless part of π_{ij} to vanish. The momentum constraints $\mathcal{H}^i = 0$ then determine α to be constant, and the Hamiltonian constraint $\mathcal{H}=0$ fixes h_{ij} to be a constant negative curvature metric, with $R=-2\alpha^2$. By rescaling the coordinates, we can fix $R=-1$. The boundary conditions (3.2) thus determine the set of metrics of constant curvature -1, which are well known to parametrize the Teichmüller space of Σ.

To define the gravitational path integral, we must specify boundary conditions which fix half of the boundary data for g_{ab}. The conditions (3.2) clearly do this, as can be seen in more detail in the general analysis of the canonical structure of $(2+1)$-dimensional gravity in Ref. 14. The path integral then reduces to an integral over the Teichmüller space of metrics on Σ. But as we have already observed, this Teichmüller space is precisely the space of complex structures appearing in the WZNW action determined from the Chern-Simons theory (2.1).

Of course, in string theory one integrates over moduli space, not Teichmüller space; it is still necessary to divide out the mapping class group of Σ. This, too, can be understood from the $(2+1)$-dimensional point of view. The gravitational action (2.4) is invariant under the entire group of diffeomorphisms of $M=[0,1]\times\Sigma$. The constraints generate the diffeomorphisms isotropic to the identity, but we must still divide out the three-dimensional mapping class group, which includes[16] diffeomorphisms of the form $\mathcal{D}=1\times\mathcal{D}_0$, where $\mathcal{D}_0$ is a Dehn twist of Σ. Dividing out the three-dimensional mapping class group of $[0,1]\times\Sigma$ thus requires dividing out the two-dimensional mapping class group of Σ, reducing our integral to one over the moduli space of Σ.

Note that if we had a chosen a different topology for M, this would no longer be the case. A handle-body (that is, a solid genus-g surface), for instance, has as its symmetries only a subgroup of the mapping class group,[17] and corresponding chiral WZNW model would thus be integrated over a space larger than the genus-g moduli space.

Although our discussion of constraints and boundary conditions has been for pure gravity, the addition of the Chern-Simons action (2.1) does not affect the conclusions, since S_1 does not contain the metric. The F^2 action (2.2) does contain g_{ab}, of course, and will alter the constraints (3.1). For the $E=0$ states, however, $F_{ij}=0$, and it is easily checked that the momentum constraint is unaffected. The Hamiltonain constraint will be slightly altered, so the scalar curvature ${}^{(2)}R$ will only be constant up to terms of order $1/\gamma$; but this merely provides us with a slightly different model for Teichmüller space, and will not change the final results.

(4) The gravitational Chern-Simons term and the central charge.—It remains for us to consider the gravitational Chern-Simons action (2.5). The Chern-Simons gauge theory action (2.1) induces a gravitational Chern-Simons term, whose coefficient can be calculated either directly from the perturbation expansion[18] or by index theorem arguments.[1,19] For a $\mathrm{U}(1)^n$ Chern-Simons theory, the coefficient is $k'=in/24$. For a non-Abelian theory, general arguments[20] suggest that the coefficient should become $k'=ic/24$, where c is the central charge of the corresponding WZNW model. It would be interesting to check this result directly in perturbation theory.

In Euclidean space, (2.5) would be a Chern-Simons term for an SO(3) gauge theory, and invariance under large gauge transformations would require k' to be an integer. For Lorentzian metrics, this is not the case $-\pi_3(\mathrm{SO}(2,1))=0$, and there are no gauge transformations with nonzero winding numbers. But SO(2,1) contains a U(1) subgroup, and it is known that the coefficient of a U(1) Chern-Simons theory must also be quantized.[2,21] The induced gravitational Chern-Simons term thus gives the three-dimensional version of the two-dimensional conformal anomaly.

(5) Conclusion.—Two-dimensional conformal field theories arise naturally from three-dimensional Chern-Simons theories. We have now taken a step towards showing that more is true: Full string-theory amplitudes, integrated over moduli space, can arise from three- dimensional Chern-Simons theories coupled to gravity. It may thus be possible to reinterpret string theory as a three-dimensional topological theory.

Of course, many questions remain. The boundary conditions (3.2) were chosen fairly arbitrarily in order to get the right answer. It would be interesting to see if they arise from a higher-derivative theory in the same way that the Chern-Simons boundary conditions $\Phi=\Phi[A_{\bar{z}}]$ do. We do not yet understand the b-c ghost system of string theory from a three-dimensional point of view; a preliminary calculation of the $(2+1)$-dimensional graviational path integral seems to give no corresponding Faddeev-Popov determinant. Further, the coupling of gravity to the Chern-Simons theory only through the regulator term (2.2) may not be completely satisfying, and it is important to check that our results do not depend on the choice of regularization. But although these questions are not yet answered, we believe there is a reasonable chance of finding a complete three-dimensional formulation of string theory.

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