Anomalous Dimensions and the Renormalization Group in a Nonlinear Diffusion Process

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We present a renormalization-group (RG) approach to the nonlinear diffusion process $\partial_t u = D\partial_x^2 u$, with $D=\frac{1}{2}$ for $\partial_x^2 u > 0$ and $D=(1+\epsilon)/2$ for $\partial_x^2 u < 0$, which describes the pressure during the filtration of an elastic fluid in an elastoplastic porous medium. Our approach recovers Barenblatt's long-time result that, for a localized initial pressure distribution, $u(x,t) \sim t^{-(\alpha+1/2)} f(x/\sqrt{t},\epsilon)$, where f is a scaling function and $\alpha = \epsilon/(2\pi e)^{1/2} + O(\epsilon^2)$ is an anomalous dimension, which we compute perturbatively using the RG. This is the first application of the RG to a nonlinear partial differential equation in the absence of noise.

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Buckingham's Π theorem[1] states that the dependence of a physical quantity on a set of dimensionful parameters may be expressed as the dependence of a dimensionless quantity Π on dimensionless combinations $\Pi_0, \Pi_1, \ldots, \Pi_n$ of the governing parameters:

$$\Pi = f(\Pi_0, \Pi_1, \ldots, \Pi_n) . \tag{1}$$

This elementary result is the root of dimensional analysis, and, as is well known, has an extraordinary range of applications.[2] Of particular importance is the application of this result to the solution of partial differential equations (PDE's) describing *inter alia* the time-dependent behavior of a system away from equilibrium. Examples of current interest are the approach to equilibrium of a binary system quenched into the two-phase region of the phase diagram,[3] diffusion-controlled crystal growth,[4] and hard thermal turbulence.[5]

In these and other examples, one is typically interested in the limit when one of the dimensionless variables, Π_0, tends to zero; here we will focus on the case in which this corresponds to the long-time behavior of the system of interest. For example, in spinodal decomposition,[3] as time $t \rightarrow \infty$, the structure factor S at time t as a function of wave number k obeys the scaling law $S(k,t) = l(t)^d \times F(kl(t))$, with $l(t) \sim t^{\phi}$. In velocity-selection problems, such as those related to dendritic crystallization,[6] the steady-state solution of an evolution equation is of the form $u(x,t) = f(x - vt)$. Transforming $x = \ln X$, $t = \ln T$, and $u(x,t) = U(X,T)$, this solution can be expressed in the form $U(X,T) = F(XT^{-v})$. The central problems are (1) to evaluate the exponents appearing in these solutions, such as ϕ and v in the examples above, and (2) to account for the stability of the scaling form.

In many cases, Buckingham's Π theorem is sufficient to solve the first of these problems. However, as has been discussed at length by Barenblatt,[7] there is an even larger category of situations where simple dimensional analysis fails: the function f in Eq. (1) is not well defined in the limit $\Pi_0 \rightarrow 0$. Instead, the following limit is well defined:

$$\lim_{\Pi_0 \to 0} \frac{\Pi}{\Pi_0^{\alpha}} = \lim_{\Pi_0 \to 0} \Pi_0^{-\alpha} f\left(\Pi_0, \frac{\Pi_1}{\Pi_0^{\alpha_1}}, \ldots, \frac{\Pi_n}{\Pi_0^{\alpha_n}}\right), \tag{2}$$

with the exponents $\alpha, \alpha_1, \ldots, \alpha_n$ being real parameters, not determined by dimensional analysis, but determined, in principle, by the differential equation obeyed by f. In the example of velocity selection in dendritic growth, the velocity corresponds to an exponent of the above type, and is determined by a solvability condition.[6]

The purpose of this Letter is to show explicitly that these exponents are nothing other than the anomalous dimensions of field theory,[8] and can be computed using the techniques of the renormalization group (RG). We illustrate our ideas by solving a nonlinear diffusion equation, discussed by Barenblatt and Sivashinsky,[7,9] using renormalized perturbation theory and comparing with the exact solution. To our knowledge, this is the first time that the renormalization group has been used in this way. We believe that methods similar to those used here will be useful in studying the more interesting physical situations mentioned earlier.[10]

We start with the one-dimensional nonlinear diffusion problem[7]

$$\partial_t(x,t) = D\,\partial_x^2 u(x,t) , \tag{3}$$

where $D=\frac{1}{2}$ for $\partial_x^2 u > 0$ and $D=(1+\epsilon)/2$ for $\partial_x^2 u < 0$. This equation, hereafter referred to as Barenblatt's equation, describes the filtration of a compressible fluid through an elastic porous medium which is irreversibly deformable.[7] It is readily verified that, with the choice of diffusion coefficient given above, the sum of two solutions to Barenblatt's equation does not constitute a third solution, showing that the equation is indeed nonlinear. The value of ϵ is determined by the elastic constants of the fluid and the porous medium. We consider the case when the initial condition is given by

$$u(x,0)=g(x)\equiv\frac{Q_0}{(2\pi l^2)^{1/2}}\exp\left[-\frac{x^2}{2l^2}\right] \quad (4)$$

and we seek solutions which vanish at infinity. When $\epsilon=0$, the equation is simply the diffusion equation; with the stated initial conditions, the solution at large times $t\gg l^2/D$ is of the form

$$\lim_{t\to\infty} u(x,t)=t^{-1/2}f(x/\sqrt{t}) \quad (\epsilon=0)\,, \quad (5)$$

where the scaling function f is twice differentiable. Notice that l is absent from Eq. (5). In fact, the limit of Eq. (5) can also be attained by keeping t fixed, solving the initial-value problem, and then taking the limit $l\to 0$.

Now consider the case when $\epsilon\neq 0$. The natural extension of Eq. (5) is that

$$\lim_{t\to\infty} u(x,t)=t^{-1/2}f(x/\sqrt{t},\epsilon) \quad (\epsilon\neq 0)\,, \quad (6)$$

where the scaling function f is twice differentiable. Remarkably, this is not the case. Substituting the *Ansatz* Eq. (6) into Barenblatt's equation, one obtains two ordinary differential equations in the variable $\xi\equiv x/\sqrt{t}$. One equation is valid for $\partial_x^2 u>0$, while the other is valid when $\partial_x^2 u<0$. As shown in Ref. 7, it is not possible to match the solutions across the point where $\partial_x^2 u=0$. A general theorem due to Kamenomostskaya[11] proves that the initial-value problem has a unique solution with continuous derivatives in x up to second order; hence we conclude that there is no nontrivial solution to Barenblatt's equation of the form of Eq. (6).

To investigate the actual form of the solution of Barenblatt's equation, we construct a perturbation theory in ϵ about the conventional linear diffusion equation. *A priori*, one might reasonably expect a naive expansion to fail, given that the introduction of ϵ has apparently made a qualitative change to the solutions. Nevertheless, the breakdown of the perturbation expansion is controllable using the renormalization group. The formal solution to Eq. (3) is

$$u(x,t)=\int dy\, G(x-y,t)g(y)+\frac{\epsilon}{2}\int_0^t ds\int dy\, G(x-y,t-s)\theta[-\partial_s u(y,s)]\partial_y^2 u(y,s)\,, \quad (7)$$

where θ is the Heaviside step function and G is the Green's function

$$G(x,t)=\frac{1}{(2\pi t)^{1/2}}\exp\left[-\frac{x^2}{2t}\right]. \quad (8)$$

On dimensional grounds, u depends on $x/\sqrt{t}$, $l/\sqrt{t}$, and ϵ. Writing

$$u(x,t)=u_0(x,t)+\epsilon u_1(x,t)+\cdots\,, \quad (9)$$

we find that the zeroth-order term is simply

$$u_0=\frac{Q_0}{[2\pi(t+l^2)]^{1/2}}\exp\left[-\frac{x^2}{2(t+l^2)}\right]. \quad (10)$$

The first-order term is calculated straightforwardly using the zeroth-order solution in the Heaviside function. As anticipated, u_1 diverges as $t\to\infty$, or equivalently, as $l\to 0$. We find that

$$u_1=\frac{Q_0}{4\pi\sqrt{t}}e^{-x^2/2t}\int_{l^2}^t\frac{ds}{s}\int_{-1}^1 dw\, e^{-w^2/2}(w^2-1) + \text{regular terms}+O(l^2)\,, \quad (11)$$

so that the singular part of u_1 is given by

$$u_1^s=-\frac{1}{(2\pi e)^{1/2}}\ln\left[\frac{t}{l^2}\right]u_0(x,t)\,. \quad (12)$$

Thus, the bare-perturbation-theory result is

$$u(x,t)=\frac{Q_0}{(2\pi t)^{1/2}}e^{-x^2/2t}\left[1-\frac{\epsilon}{(2\pi e)^{1/2}}\ln\left[\frac{t}{l^2}\right]+O(\epsilon^2)\right]+\text{nonsingular terms}\,. \quad (13)$$

We chose to treat the divergence of the bare perturbation theory by regarding l as a regularization parameter. The theory can be rendered finite in the limit $l\to 0$ by introducing the renormalized variable

$$u_R(x,t)=Z(l/\mu)u(x,t)\,, \quad (14)$$

where the subscript R denotes the renormalized quantity. The renormalization constant Z is introduced to absorb the divergences as $l\to 0$, and so depends upon l and ϵ. Z being dimensionless, we must also introduce a new quantity with the dimensions of length, which is denoted by μ. The renormalized variable u_R is independent of l, and so is finite as $l\to 0$. We proceed by expanding $Z=\sum_{n=0}^\infty a_n(l/\mu)\epsilon^n$ with $a_0=1$. The coefficients a_n for $n\geq 1$ are determined order by order in ϵ in such a way that the divergences in u_l are cancelled out. To $O(\epsilon)$, we obtain

$$a_1\left[\frac{l}{\mu}\right]=\frac{\ln(C_1\mu^2/l^2)}{(2\pi e)^{1/2}}\,, \quad (15)$$

where C_1 is an arbitrary constant, which will turn out to be unimportant. Thus

$$u_R(x,t)=\frac{\mu e^{-x^2/2t}}{(2\pi t)^{1/2}}\left[1-\frac{\epsilon}{(2\pi e)^{1/2}}\ln\left(\frac{t}{C_1\mu^2}\right)+O(\epsilon^2)\right]. \tag{16}$$

Notice that this expression for u_R is manifestly finite as $l\rightarrow 0$. The length scale μ is still undetermined, and indeed is arbitrary; this is essential for the following steps.

At this point, we need to make contact with the initial-value problem: Eq. (16) apparently describes a family of solutions. We choose a particular solution by insisting that, e.g., at the origin at some given time t^*, $u_R(0,t^*)$ has the value $Q(t^*)$. Then the corresponding particular solution is

$$u^*(x,t)=Q(t^*)\frac{u_R(x,t)}{u_R(0,t)}. \tag{17}$$

The requirement that u_R has a certain value at a certain time and place is enough to determine all the arbitrary constants $C_1,\ldots$ which are introduced during the renormalization procedure. It is straightforward to verify that this occurs to $O(\epsilon)$, and we obtain

$$u_R^*(x,t)=Q(t^*)\left(\frac{t^*}{t}\right)e^{-x^2/2t}\times\left[1-\frac{\epsilon}{(2\pi e)^{1/2}}\ln\left(\frac{t}{t^*}\right)+O(\epsilon^2)\right]. \tag{18}$$

This perturbative expression is valid at best when $t\sim t^*$; otherwise, the terms in the perturbation series are not small. However, we have not yet made a specific choice of t^* and $Q(t^*)$. This arbitrariness corresponds to the arbitrariness in choosing the length scale μ, which has now disappeared from the expression for u_R. We can exploit this arbitrariness as follows: Suppose that we wanted to know the behavior of the solution with given initial conditions at a time of, e.g., 1×10^6 sec. If we knew the value of Q at time $t^*=5$ sec, then we could, in principle, use the perturbation series (18) to obtain the required value. The result would, of course, be meaningless, because of the poor convergence of the perturbation series. On the other hand, if we knew the value Q at a time T^{**} close to 1×10^6 sec, then the series would indeed be useful in estimating the required value of u_R. The important point is that we require that the value $Q(t^{**})$ be that which we would have obtained if we had solved the initial-value problem with $u_R(0,t^*)=Q(t^*)$, found the value of $u_R(0,t^{**})$, and then set this equal to $Q(t^{**})$.

This desideratum can be achieved using the renormalization-group argument of Gell-Mann and Low.[12] The function u_R^* represents the solution of the specific initial-value problem of Barenblatt's equation which has the value at the origin of Q^* when $t=t^*$. The actual solution does not depend upon the choice of the time t^*. Thus

$$\frac{du_R^*(x,t)}{dt^*}=\frac{\partial u_R^*(x,t)}{\partial t^*}+\frac{\partial u_R^*(x,t)}{\partial Q}\frac{\partial Q}{\partial t^*}=0. \tag{19}$$

The quantity $\beta\equiv t^*\partial Q/\partial t^*$ is analogous to the β function in field theory: It describes how the value of Q varies as t^* is varied, so that the function u_R^* remains unchanged as the solution of the specific initial-value problem of Barenblatt's equation.

We can evaluate $\beta(Q)$ perturbatively from the expression u_R^*. We obtain

$$\beta=-Q\left(\frac{1}{2}+\frac{\epsilon}{(2\pi e)^{1/2}}\right). \tag{20}$$

Integrating this equation, we obtain

$$Q(t^*)=(At^*)^{-[1/2+\epsilon/(2\pi e)^{1/2}]}, \tag{21}$$

where A is a constant of integration determined by the initial conditions. Finally, we insert this value into Eq. (18) and set $t^*=t$, which we are entitled to do. Hence we obtain

$$u_R(x,t)=\frac{A}{t^{1/2+\alpha}}e^{-x^2/2t} \tag{22}$$

with the anomalous dimension $\alpha=\epsilon/(2\pi e)^{1/2}+O(\epsilon^2)$. We have verified that this result agrees with the exact result reported in Ref. 7. Furthermore, we have extended the perturbation-theory calculation to second order in ϵ, verifying that the logarithms in the perturbation series do indeed sum up the way that the RG predicts, and finding that the coefficient of the ϵ^2 term in α is approximately -0.102.

We have shown that the asymptotic behavior of u changes from $u\sim t^{-1/2}$ to $t^{-(1/2+\alpha)}$ upon the introduction of the nonlinearity into the diffusion equation. The anomalous dimension α here has an origin similar to that at the critical point of a field theory: Even though the characteristic length scale (the width of the diffusion distribution in one case, the correlation length in the other) is large compared to the microscopic cutoff (the initial distribution width in one case, the lattice spacing in the other), the latter cannot simply be set to zero, because the theory is not well defined in this limit. Consequently, the microscopic length must be included in the dimensional analysis: The values of the (critical) exponents differ from those predicted without taking into account the microscopic length scale.

We conclude with some remarks and speculations. First, the example presented here is important because it demonstrates that the RG can be applied to PDE's without adding a noise source. Noise is believed to be irrelevant during the scaling regime of spinodal decomposition; its inclusion in current RG approaches to the problem of turbulence[13] is an unsatisfactory feature.

Second, our work establishes a connection between intermediate asymptotics[7] and the RG, which is explained in more detail elsewhere.[14] The exponents $\{\alpha\}$ in Eq. (2) are in general computable as the solutions of a nonlinear eigenvalue problem.[6,7] Are there analogous equations for critical exponents in general? Finally, our calculation can be interpreted from the Wilson formulation of the RG.[15] There are close similarities between our work and the method of Lie groups.[16] We anticipate that this will prove useful in numerical calculations; in particular, we hope that work along these lines will facilitate a systematic development of qualitative numerical methods, such as the cell dynamic schemes which have proved useful in studying phase separation[17] and solidification.[18]

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