Resummation and Gauge Invariance of the Gluon Damping Rate in Hot QCD

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In high-temperature QCD, an infinite set of higher-loop diagrams must be resummed in order to compute the gluon damping rates to leading order in the coupling constant. After resummation, the damping rates, as determined by the positions of the poles in the gluon propagator, are gauge invariant.

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Recently, there has been much controversy concerning the damping rates in perturbative QCD at high temperature.[1,2] Attention has focused on the plasmon, which is a collective, longitudinal mode of the gluon. Explicit calculations appear to show that the magnitude and even the sign of its damping rate is gauge dependent. Both the plasmon and the transverse modes of the gluon are physical degrees of freedom, and their damping rates are determined by the position of the poles in the gluon propagator. These rates are physical quantities, and must be gauge invariant if computed correctly.

The apparent gauge dependence occurs because the calculations done to date are incomplete—there are diagrams of higher order in the loop expansion which contribute to the same order in the coupling constant g as the one-loop diagram.[2,3] As suggested in Ref. 4, these higher-order effects can be systematically resummed into effective propagators and vertices. In this Letter, we prove that this resummation cures the problem of the gauge dependence of the damping rates: When all terms of leading order in g are computed by using effective propagators and vertices, the gluon damping rates are gauge invariant. This work is an application of the general program of resummation developed in Ref. 5.

We use the imaginary-time formalism to compute at a nonzero temperature T. In momentum space the propagator for a scalar boson is $\Delta(P)=1/P^2$, $P^\mu=(p^0,\bar{p})$, where $P^2=(p^0)^2+p^2$, and in imaginary time $p^0=2\pi jT$ for integral j. Amplitudes in real time are obtained from those in imaginary time by analytically continuing the discrete p^0 to a continuous Minkowski energy, $p^0=-i\times\omega$.

At high temperature, the average momentum of massless fields is of order T; through interactions, these fields acquire "masses" of order gT. After analytic continuation, a momentum is "soft" if all of its components, ω and $\bar{p}$, are of order gT, and "hard" if any component is of order T. The distinction between hard and soft momenta is crucial, for it separates amplitudes which require resummation from those for which ordinary perturbation theory applies. For amplitudes with hard external legs, perturbative corrections are uniformly down by at least one power of g. In contrast, if every external leg of an amplitude is soft, there are loop corrections which are g^2T^2/P^2 times the corresponding tree amplitude, where P is a typical soft external momentum. Since P is of order gT, these corrections are as important as the tree amplitude. We call such corrections "hard thermal loops," for they arise from diagrams with hard loop momentum and are due solely to thermal fluctuations.[4]

The simplest example of a hard thermal loop is from the tadpole diagram, which is proportional to the integral $\mathrm{Tr}\Delta(K)\approx T^2/12$. The other integrals which generate the hard thermal loops in N-gluon amplitudes are of the form

$$\mathcal{J}^{\mu_1\cdots\mu_N}(P_1\cdots P_N)\approx\mathrm{Tr}\,K^{\mu_1}\cdots K^{\mu_N}\Delta(P_1-K)\cdots\Delta(P_N-K)\,. \tag{1}$$

K^μ is the loop momentum, with $\mathrm{Tr}=T\sum_{k^0}\int d^3k/(2\pi)^3$. The hard thermal loop in the integral is the term proportional to T^2; we use the symbol $\approx$ to indicate that the function $\mathcal{J}^{\mu_1\cdots\mu_N}$ includes just this part of the integral.

Before discussing general properties of hard thermal loops, we list those which are needed for the gluon damping rate at leading order. We assume an $\mathrm{SU}(N_c)$ gauge theory with N_f flavors of massless quarks in the fundamental representation. The hard thermal loop in the gluon self-energy $\delta\Pi^{\mu\nu}$ was first calculated by Silin and co-workers,[6-8]

$$\delta\Pi^{\mu\nu}(P)=4g^2(N_c+\tfrac{1}{2}N_f)[\mathcal{J}^{\mu\nu}(0,P)-\tfrac{1}{2}\delta^{\mu\nu}\mathrm{Tr}\Delta(K)]\,. \tag{2}$$

The contribution of the quark loop involves fermionic propagators, but hard thermal loops obey identities[5] which allow this to be rewritten entirely in terms of integrals over bosonic propagators; these identities are used below in (3) and (4) as well.

Denoting the three-gluon amplitude by $-igf^{abc}$

$\times\Gamma^{\mu\nu\lambda}(P,Q,R)$, the hard thermal loop is

$$\delta\Gamma^{\mu\nu\lambda}(P,Q,R) = -8g^2(N_c+\tfrac{1}{2}N_f)\mathcal{J}^{\mu\nu\lambda}(0,P,-Q). \quad (3)$$

For the four-gluon vertex we only need the amplitude traced over two color indices. Denoting this component by $-g^2N_c\delta^{ab}\Gamma^{\mu\nu\lambda\sigma}(P,Q,R,S)$, the hard thermal loop is

$$\delta\Gamma^{\mu\nu\lambda\sigma}(P,Q,R,S) = 16g^2(N_c+\tfrac{1}{2}N_f) \times\mathcal{J}^{\mu\nu\lambda\sigma}(0,P,P+R,-S). \quad (4)$$

In Ref. 5 we show that hard thermal loops arise only in the amplitudes between N gluons or between a quark pair and $N-2$ gluons. They possess several remarkable properties.

Hard thermal loops arise just from one-loop subdiagrams.— At soft P one-loop amplitudes develop terms proportional to T^2, but never to higher powers of T. All one-particle irreducible diagrams at higher-loop order are smaller by powers of g. Hard thermal loops arise just from the region of integration in which the loop momentum K is hard, from diagrams with sufficiently many powers of K in the numerator. Diagrams with external ghost lines do not have hard thermal loops, because the relevant integrals always have one less power of K than those in (1).

Hard thermal loops are gauge invariant.— Klimov and Weldon[8] showed that the hard thermal loops in the self-energies are the same in any covariant gauge. We have proven that for *arbitrary* amplitudes, the hard thermal loops are the same in *any* covariant or Coulomb-type gauge:[5] On these grounds we assume that this holds in all gauges. This gauge invariance is unexpected, for the external momenta need only be soft, and do not have to be on the mass shell.

Hard thermal loops satisfy simple Ward identities. —The Ward identity for the hard thermal loop in the gluon self-energy shows that it is transverse: $P^\mu\delta\Pi^{\mu\nu}(P)=0$. The Ward identities for the hard thermal loops in the three- and four-gluon vertices are

$$R^\lambda\delta\Gamma^{\mu\nu\lambda}(P,Q,R) = \delta\Pi^{\mu\nu}(P)-\delta\Pi^{\mu\nu}(Q),$$
$$S^\sigma\delta\Gamma^{\mu\nu\lambda\sigma}(P,Q,R,S) = \delta\Gamma^{\mu\nu\lambda}(P+S,Q,R) - \delta\Gamma^{\mu\nu\lambda}(P,Q+S,R). \quad (5)$$

The need for resummation in calculations involving soft momenta is apparent, for when the external momenta are soft, hard thermal loops are as large as the tree amplitude.[4] We therefore define effective propagators and vertices which resum all hard thermal loop corrections to the bare amplitude. We denote effective propagators and vertices by a left superscript *.

The effective gluon propagator resums all insertions of the hard thermal loop $\delta\Pi$ in the gluon self-energy. The transverse part of the effective inverse propagator is ${}^*\Delta_{\mu\nu}^{-1}(P)=P^2\delta^{\mu\nu}-P^\mu P^\nu-\delta\Pi^{\mu\nu}(P)$, where $\delta\Pi^{\mu\nu}$ is given in (2). ${}^*\Delta^{-1}$ is transverse because of the Ward identify satisfied by $\delta\Pi^{\mu\nu}$. The effective propagator is obtained by adding a gauge fixing term to ${}^*\Delta^{-1}$ and inverting. The difference between the effective gluon propagator in covariant gauge ${}^*\Delta^{\mu\nu}$ and that in Coulomb gauge ${}^*\Delta_C^{\mu\nu}$ has the form

$${}^*\Delta^{\mu\nu}(K)-{}^*\Delta_C^{\mu\nu}(K)=K^\mu K^\nu\,{}^*\Delta_1(K) + (n^\mu K^\nu+K^\mu n^\nu)\,{}^*\Delta_2(K), \quad (6)$$

where $n^\mu K^\mu=k^0$. For example, in the Feynman gauge, ${}^*\Delta_1(K)=k_0^2\,{}^*\Delta_l(K)/(K^2)^2$ and ${}^*\Delta_2(K)=-k^0\,{}^*\Delta_l(K)/K^2$, where ${}^*\Delta_l(K)=1/[k^2-\delta\Pi^{00}(K)]$. The covariant propagator satisfies

$${}^*\Delta_{\mu\lambda}^{-1}(K)\,{}^*\Delta^{\lambda\nu}(K)=\delta^{\mu\nu}-K^\mu K^\nu/K^2, \quad (7)$$

$${}^*\Delta_{\mu\nu}^{-1}(K)n^\nu\,{}^*\Delta_l(K)=(K^2n^\mu-k^0K^\mu)/k^2. \quad (8)$$

These identities are helpful because while effective quantities enter into the left-hand sides, the right-hand sides are free of them.

The effective vertices are formed by adding the hard thermal loop to the bare vertex: ${}^*\Gamma=\Gamma+\delta\Gamma$. The Ward identities satisfied by the effective vertices follow immediately from (5). Because of the simplicity of (5), they are *identical* in form to the Ward identities for the bare vertices. There are no hard thermal loops for amplitudes with external ghosts, so the ghost propagator and the ghost-gluon vertex remain the same as in the bare expansion.

To calculate systematically, loop integrals must be separated into integrals over hard and over soft momenta.[5] Soft lines require effective propagators, while bare propagators are used for hard lines. If all the legs of a vertex are soft, an effective vertex is needed; otherwise a bare vertex suffices. In the resulting diagrammatic expansion, only a finite number of diagrams contribute to any fixed order in g.

We apply the effective expansion to the gluon self-energy at soft external momentum. We define the effective self-energy for the gluon ${}^*\Pi^{\mu\nu}$ to be the one-particle irreducible correction to the effective propagator ${}^*\Delta^{\mu\nu}$. The three diagrams with soft-loop momenta which contribute to ${}^*\Pi$ at order g, ${}^*\Pi={}^*\Pi_{3g}+{}^*\Pi_{4g}+{}^*\Pi_{gh}$, are shown in Fig. 1. In covariant gauge, they are

$${}^*\Pi_{3g}^{\mu\nu}(P)=\tfrac{1}{2}g^2N_c\,\mathrm{Tr}_{(\mathrm{soft})}\,{}^*\Gamma^{\sigma\mu\lambda}(-P+K,P,-K)\,{}^*\Delta^{\lambda\lambda'}(K)\,{}^*\Gamma^{\lambda'\nu\sigma'}(-K,P,-P+K)\,{}^*\Delta^{\sigma'\sigma}(P-K), \quad (9)$$

$${}^*\Pi_{4g}^{\mu\nu}(P)=-\tfrac{1}{2}g^2\,\mathrm{Tr}_{(\mathrm{soft})}\,{}^*\Gamma^{\mu\nu\lambda\sigma}(P,-P,K,-K)\,{}^*\Delta^{\lambda\sigma}(K), \quad (10)$$

$${}^*\Pi_{gh}^{\mu\nu}(P)=g^2N_c\,\mathrm{Tr}_{(\mathrm{soft})}K^\mu(P-K)^\nu\Delta(K)\Delta(P-K). \quad (11)$$

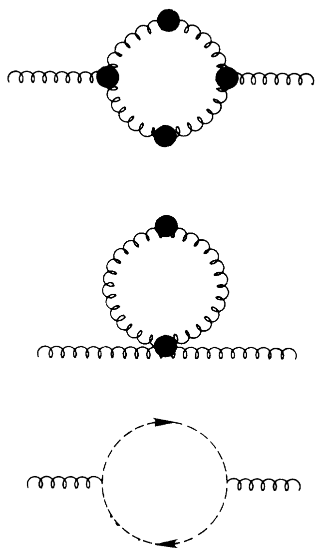

FIG. 1. One-loop diagrams that contribute to the effective gluon self-energy ${}^*\Pi^{\mu\nu}$. The solid circles represent effective propagators and vertices.

The gluon loops in (9) and (10) differ from those in the bare expansion only in that everything is "starred": bare propagators and vertices are replaced by effective quantities. The ghost loop in (11) is the same as in the bare expansion. In (9)-(11), $\mathrm{Tr}_{(\mathrm{soft})}$ indicates that the integral only runs over soft k. For kinematic reasons, at leading order this is the only region which contributes to the discontinuity of ${}^*\Pi$ on the mass shell and hence to the damping rates.[5]

In Coulomb gauge the effective self-energy to order g is ${}^*\Pi_C = {}^*\Pi_{C,3g} + {}^*\Pi_{C,4g} + {}^*\Pi_{C,gh}$. The terms from a gluon loop, ${}^*\Pi_{C,3g}$ and ${}^*\Pi_{C,4g}$, are obtained by replacing each covariant propagator ${}^*\Delta^{\mu\nu}$ in (9) and (10) with Coulomb propagators ${}^*\Delta^{\mu\nu}_C$. The Coulomb ghost contribution ${}^*\Pi_{C,gh}$ is the same as in the bare expansion.

For a physical mode, the position of the pole in a propagator is a gauge-invariant quantity. At lowest order, the poles in the effective gluon propagator are determined by the mass-shell condition

$$ {}^*\Delta^{-1}_{\mu\nu}(P)e^{\nu}_i(P) = 0\,, \tag{12}$$

where $e^\nu_i(P)$ is the gluon wave function. At nonzero temperature, there are three physical modes labeled by the index i: the two transverse gluons and the longitudinal plasmon. Since the hard thermal loop in the gluon self-energy is gauge invariant, so is ${}^*\Delta^{-1}$, and hence the mass shells for transverse gluons and plasmons. Their detailed forms are given in Refs. 6-8. The gluon wave functions depend upon the gauge; in Coulomb and covariant gauge, they are related by

$$ e^{\mu}_{C,i}(P) = e^{\mu}_i(P) - P^\mu \mathbf{p}\cdot\mathbf{e}_i(P)/p^2\,. \tag{13}$$

Beyond leading order, positions of the poles are determined by the zeros of the effective inverse propagator ${}^*\Delta^{-1}_{\mu\nu} - {}^*\Pi^{\mu\nu}$. If all one wants is the shift in the mass-shell condition to order g, we can take a shortcut. Construct a "two-gluon $\mathcal{T}$ matrix" by sandwiching the effective self-energy between physical wave functions: in covariant gauge, $\mathcal{T}_{ij} = e^\mu_i\, {}^*\Pi^{\mu\nu} e^\nu_j$. (Henceforth we suppress the dependence of $\mathcal{T}_{ij}$, ${}^*\Pi^{\mu\nu}$, and e^μ_i on P; $\mathcal{T}_{ij}$ and e^μ_i are of course defined with P on the mass shell.) Then it can be shown that the only terms in ${}^*\Pi^{\mu\nu}$ which shift the position of the pole are those which contribute to the $\mathcal{T}$ matrix. For small momentum, the real part of the $\mathcal{T}$ matrix gives a perturbative correction of order g to the mass shell, which is itself of order g^2T^2.[6-8] The imaginary part of the $\mathcal{T}$ matrix is of order g^3T^2, and its eigenvalues are proportional to the damping rates.

To prove that the damping rates are gauge invariant, we show that the contribution from soft-loop momenta to the $\mathcal{T}$ matrix is the same in covariant gauge as in the Coulomb gauge:

$$ \mathcal{T}_{C,ij} \equiv e^{\mu}_{C,i}\, {}^*\Pi^{\mu\nu}_C e^{\nu}_{C,j} = e^\mu_i\, {}^*\Pi^{\mu\nu}_C e^\nu_j = e^\mu_i\, {}^*\Pi^{\mu\nu} e^\nu_j \equiv \mathcal{T}_{ij}\,. \tag{14}$$

The proof proceeds in two steps. First, we derive the Ward identity for the effective self-energy: $P^\mu\, {}^*\Pi^{\mu\nu}P^\nu = 0$ for arbitrary P. To demonstrate this, the Ward identities are used to reduce the contraction of P^μ with the effective three- or four-gluon vertices in ${}^*\Pi_{3g}$ and ${}^*\Pi_{4g}$. For instance, P^μ contracted with the effective three-gluon vertex ${}^*\Gamma^{\mu\nu\lambda}$ is equal to ${}^*\Delta^{-1}_{\nu\lambda}(K)$ minus ${}^*\Delta^{-1}_{\nu\lambda}(P-K)$. After using (7) followed by the Ward identities again, we find terms proportional to ${}^*\Delta^{-1}_{\nu\lambda}(P)$, which vanish upon contraction with P^ν. The remaining terms in $P^\mu({}^*\Pi^{\mu\nu}_{3g} + {}^*\Pi^{\mu\nu}_{4g})P^\nu$ reduce to an expression free of effective propagators and vertices. This cancels against the contribution of the ghost loop, $P^\mu\, {}^*\Pi^{\mu\nu}_{gh}P^\nu$. In this way it can also be shown that the effective self-energy in Coulomb gauge is transverse, $P^\mu\, {}^*\Pi^{\mu\nu}_C P^\nu = 0$, and that it obeys $P^\mu\, {}^*\Pi^{\mu\nu}_C e^\nu_i = 0$ on the mass shell. Using the relation between Coulomb and covariant wave functions in (13) establishes the first equality in (14).

To prove the second equality in (14), we use (6)-(8) and (12) and the effective Ward identities to show that

$$ e^\mu_i({}^*\Pi^{\mu\nu}_{3g} + {}^*\Pi^{\mu\nu}_{4g})e^\nu_j - e^\mu_i({}^*\Pi^{\mu\nu}_{C,3g} + {}^*\Pi^{\mu\nu}_{C,4g})e^\nu_j $$

reduces to a form free of effective quantities. This cancels exactly against the difference between the ghost loops, $e^\mu_i\, {}^*\Pi^{\mu\nu}_{gh}e^\nu_j - e^\mu_i\, {}^*\Pi^{\mu\nu}_{C,gh}e^\nu_j$, completing the proof that $\mathcal{T} = \mathcal{T}_C$. Hence the gluon damping rates, for both the plasmon and the transverse modes, are gauge invariant to leading order in g.

As at zero temperature, the proof is really just an exercise in using the Ward identities repeatedly. In this regard, we remark that it is crucial that the effective expansion includes both effective propagators *and* vertices, for only then are the effective Ward identities in (5) satisfied.

It is also possible to prove that the damping rates are positive.[5] In Coulomb gauge, the only states which contribute to discontinuities have positive weight. Using this, a diagrammatic analysis shows that in Coulomb gauge the sum of all cuts through the diagrams of Fig. 1 can be written as a sum of amplitudes squared; these cuts run both through effective propagators *and* vertices.

In previous work,[1,2] the plasmon damping rate was calculated using one-loop diagrams constructed out of bare propagators and vertices. The first explicit calculations were by Kajantie and Kapusta,[2] who found a positive damping rate in axial gauges. Earlier, Kalashnikov and Klimov[1] pointed out that at one-loop order the damping rate is gauge dependent in covariant gauges. The problem was not taken seriously until the work of Parikh, Siemens, and Lopez:[2] they showed that the damping rate is not only gauge dependent in covariant gauges, but negative. Hansson and Zahed, and others, then attempted to overcome the gauge dependence by using a manifestly gauge-invariant formulation.[2] We have shown that this is unnecessary: Gauge invariance follows once all terms of leading order in g are included by the resummation of hard thermal loops. Hence resummation solves the longstanding problem of the gauge dependence of the plasmon damping rate.

The need for resummation was recognized in Refs. 1, 3, and 7. A complete program of resummation, which allows results to be computed consistently to leading order in g, was proposed in Ref. 4. In Ref. 5 we develop this program in detail: it applies not just to the damping rate, but to all processes involving soft momentum.

Explicit calculations of the damping rate are in progress. The results are similar to those found for a heavy fermion in Ref. 4. At zero momentum, the damping rates are a pure number times g^2T. At nonzero momentum, there is also a term g^2T times a logarithm, $\log(gT/g^2T)=\log(1/g)$. This logarithm arises from the contribution of nearly static transverse modes. It is cutoff naturally by including effects to higher order: Both the damping of the field itself, and the (nonperturbative) screening of static magnetic fields, provide a cutoff at the scale g^2T. We have also applied the effective expansion to compute the production rate for soft dileptons,[9] a quantity of direct experimental interest.

After this work was submitted for publication, two related works appeared. Kobes, Kunstatter, and Rebhan[10] extended previous proofs at $T=0$, to show that at $T\neq 0$, the physical poles in the gluon propagator have gauge-invariant positions. Our work illustrates their general arguments.

The hard thermal loops in N-gluon amplitudes were also investigated by Frenkel and Taylor.[11] Their results agree with those listed after Eq. (4). An identity of theirs was used to simplify Eq. (4). We have generalized this identity, and used it to determine the color structure of the hard thermal loops in arbitrary amplitudes.[12]

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