Unitary-Matrix Models as Exactly Solvable String Theories

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Models of unitary matrices are solved exactly in a double scaling limit, using orthogonal polynomials on a circle. Exact differential equations are found for the scaling functions of these models. For the simplest model ($k=1$), the Painlevé II equation with constant 0 is obtained. There are possible nonperturbative phase transitions in these models. The scaling function is of the form $N^{-1/(2k+1)} \times f(N^{2k/(2k+1)}(\lambda_c-\lambda))$ for the kth multicritical point. The specific heat is f^2, and is therefore manifestly positive. Equations are given for $k=2$ and 3, with a discussion of asymptotic behavior.

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Kazakov's observation[1] that critical exponents characteristic of minimal conformal matter fields occur at multicritical points of a phenomenological effective action for gravity in two dimensions has led to a nonperturbative approach[2-4] to string theory (as yet in unphysical numbers of dimensions) that holds much promise. It is, of course, widely appreciated that understanding nonperturbative aspects of string theory is a *sine qua non* for obtaining predictive physics from it. The present work may also be of interest in gauge theories, as is indicated below.

In this Letter, we solve unitary-matrix models exactly, modifying the analysis of Refs. 2-4, where Hermitian-matrix models were solved, as appropriate. The simplest of these unitary models has been studied in the planar approximation[5] in the past, in connection with the large-N approximation to QCD in two dimensions.[6] We find the nonperturbative solution of this problem, which is possibly of interest for QCD as well. Besides finding nonperturbative solutions for general actions (albeit in a certain scaling limit described below), we also find the planar behavior explicitly in the general noncritical $k=4$ model. This analysis is necessary in order to fix k parameters in the nonperturbative theory by demanding that it agree in perturbation theory with the planar result.

It is now known that the matter fields in the multicritical Hermitian models do not belong to unitary models.[7-9] The models we consider appear to lie in different universality classes from the Hermitian-matrix models. Given the drastically different large-field behaviors of these two types of models, we do not find this surprising. The positivity of the specific heat may imply that these unitary-matrix models correspond to unitary models.

We wish to study the behavior of

$$\mathcal{Z}_N \equiv \int dU \exp[-(N/\lambda)\operatorname{tr} v(U+U^\dagger)]\,,$$

where U is a $N\times N$ unitary matrix and $v(U)$ is a polynomial in U. The surface interpretation of these actions requires expanding the unitary matrices as exponentials of Hermitian matrices, but this is natural in view of the rest of the measure (see below). It is important to keep in mind that the exponential function converges absolutely on any compact domain of the complex plane, so this expansion is sensible quite independent of perturbation theory. In the present work we restrict ourselves to the simpler case of a potential invariant under complex conjugation, as indicated explicitly above. (The general case is considered in Ref. 10.) The measure is invariant under conjugation of U by another unitary matrix. Using this symmetry dU may be written (up to an irrelevant constant) as

$$dU = \prod_i d\alpha_i\, \Delta(\alpha)\bar{\Delta}(\alpha)\,,$$

where α_i are the eigenvalues of U and $\Delta\bar{\Delta}$ is the Jacobian for the change of variables,

$$\Delta \equiv \prod_{k<j} [\exp(i\alpha_k) - \exp(i\alpha_j)]\,.$$

Now $\mathcal{Z}_N$ takes the following simpler form:

$$\mathcal{Z}_N \equiv \int \prod d\alpha_i\, \Delta\bar{\Delta} \exp\left[-\frac{N}{\lambda}\sum_i v\left(z_i+\frac{1}{z_i}\right)\right],$$

where $z_i \equiv \exp(i\alpha_i)$.

The method we use to solve these models is the method of orthogonal polynomials,[11] adapted to the case of a circle. We list some properties of orthogonal polynomials on a circle, restricting to the case of actions invariant under the interchange $z \leftrightarrow 1/z$. The polynomials will be denoted $p_n(z)$, $n=0,1,\ldots$. They may be normalized so that

$$p_n(z) = z^n + \sum_{k=0}^{n-1} a_{k,n} z^k\,.$$

For measures of the restricted type that we consider, $a_{k,n}$ are real. The polynomials are orthogonal with respect to

$d\mu \equiv dz(2\pi i z)^{-1}\exp[-v(z+1/z)]$ (suppressing N/λ for the nonce):

$$\int_{\alpha=0}^{\alpha=2\pi} d\mu\, p_n(z) p_m(1/z) = h_n \delta_{n,m} .$$

As explicitly indicated, we parametrize the phase of the eigenvalues as lying in $[0,2\pi]$. $a_{0,n} \equiv R_{n-1}$ characterizes the recursion relation

$$p_{n+1}(z) = z p_n(z) + R_n z^n p_n(1/z) ,$$

and is related to h_n by $h_{n+1}/h_n = 1 - R_n^2$. A standard observation is that Z_N may be obtained as a product of the h_n,

$$Z_N \propto N! \prod h_i \propto N! \prod \left(\frac{h_i}{h_{i-1}} \right)^{N-i} \propto N! \prod (1-R_{i-1}^2)^{N-i} ,$$

which indicates why it suffices to focus on R_i. By considering $Z_{N+1}Z_{N-1}/Z_N^2$, one notes that the second derivative of the free energy F is given, in the scaling limit, by $F'' = f^2$. The function f is proportional to R_i, and the point of the following is the derivation of differential equations that determine f.

With these properties in hand, it is a simple exercise to show that

$$(n+1)(h_{n+1}-h_n) = -\int d\mu\, v'(z+1/z)(1-1/z^2) p_{n+1}(z) p_n(1/z) .$$

Using the recursion relation one derives a nonlinear functional equation for R_n. It is possible[10] to formalize this along the lines of Ref. 4, but we prefer to work out a few examples here. When $v'=1$ (the case studied in Ref. 6) we find (reinstating N/λ)

$$\lambda(n+1)R_n^2 = N R_n (R_{n+1} + R_{n-1})(1-R_n^2) .$$

Setting $n=N$, it is clear that there is a critical point when $\lambda_c = 2$, for then the roots of the limiting form of the difference equation,

$$\lambda R^2 = 2R^2(1-R^2), \quad R \equiv R_N ,$$

are degenerate. We look for a scaling solution to the difference equation we gave above by setting $R_N - R_c \equiv R_N = N^{-\mu} f(N^\nu \delta)$, where $\delta \equiv \lambda_c - \lambda$. When $\mu = \frac{1}{3}$ and $\nu = \frac{2}{3}$ we find that this is a consistent *Ansatz,* leading to $(x \equiv N^\nu \delta)$

$$-2xf + 2f^3 = f''$$

as the equation satisfied by the scaling function (with an appropriate rescaling of x). This is the Painlevé II equation. It has no critical points and its movable singularities are simple poles, as is obvious upon inspection. (The general Painlevé II equation has an arbitrary constant as well.)

As has been observed already in the Hermitian models,[2-4] the fact that this equation is second order implies that there is a parameter in the theory that cannot be determined by matching the asymptotic behavior of f to the planar approximation. The extra parameter may be identified with the position of a simple pole at a finite value of x. Brézin and Kazakov[2] identified a similar singularity with a possible condensation of handles on the world sheet. That argument does not appear to extend simply to the present case, but *if* this divergence in the specific heat signals a change in the topological structure of the world sheet, it is tempting to think of it as a deconfining transition in what is possibly the simplest model for QCD. (This is a subtle issue, discussed at length in Ref. 6.)

In the case $v'(u) = 1 + gu$ we find

$$-\lambda R_n \frac{n+1}{N} = (1-R_n^2)[-R_{n-1} - R_{n+1} + g(R_{n-2}R_{n-1}^2 + R_{n-1}^2 R_n + 2R_{n-1}R_n R_{n+1} + R_n R_{n+1}^2 - R_{n+2} - R_{n-2} + R_{n+1}^2 R_{n+2})] .$$

This leads to a quadratic equation in $R^2 \equiv R_N^2$:

$$\lambda = -2(1-R^2)(-1-g+3gR^2) .$$

For $g = -\frac{1}{4}$, this reduces to $\lambda = 3(1-R^4)/2$. The difference equation admits a scaling solution when $\mu = \frac{1}{5}$, $\nu = \frac{4}{5}$, and the differential equation (upon rescaling) is

$$-6xf + 6f^5 - 10f(f')^2 - 10f^2 f'' + f^{(4)} = 0 .$$

This equation has solutions with simple poles, but should not be compared with the analogous equations obtained by other workers[2,8] (see below).

For the case $v'(u) = 1 + g_1 u + g_2 u^2$, we do not give the explicit form of the recursion relation here because of its length. At $g_1 = -\frac{3}{7}$ and $g_2 = \frac{1}{14}$, this model is multicritical,

$$\lambda = \tfrac{10}{7}(1-R^6) .$$

The exponents are $\mu = \frac{1}{7}$, $\nu = \frac{6}{7}$, and the differential equation is

$$-\frac{10}{7}xf + \frac{10f^7}{7} - 10f^3 f'^2 - 5f^4 f'' + 5f'^2 f'' + 3ff''^2 + 4ff'f^{(3)} + f^2 f^{(4)} - \frac{f^{(6)}}{14} = 0 .$$

Again we note that solutions may have simple poles. We have no further knowledge of the character of its critical points.

In general, the differential equations are given by[10]

$$xf = \frac{2k+1}{a_k} D^{-1} \mathcal{D}^{k-1} K ,$$

where

$$a_k \equiv \frac{2k+1}{3k} a_{k-1}, \quad a_0 \equiv 1, \quad K \equiv -f'''/6 + f^2 f',$$

$$\mathcal{D} \equiv -\tfrac{1}{6} D^2 + \tfrac{2}{3} f^2 + \tfrac{2}{3} f' D^{-1} f, \quad D \equiv d/dx .$$

The operator $\mathcal{D}$ is the recursion operator for the conserved densities of the modified Korteweg-de Vries equation. These equations imply that the asymptotic behavior of f is

$$f \sim x^{1/2k} \left[1 - \frac{2k+1}{48k} x^{-(2k+1)/k} + \cdots \right], \quad x \to \infty ,$$

in powers of $x^{-(2k+1)/k}$, the string coupling constant. Note that the relative sign of the $f^{(2k)}$ and xf terms alternates.

Before comparing this asymptotic behavior to the results of Refs. 2-4, it is important to keep in mind that the relevant function is $f^2 \equiv g$. The asymptotic behavior of interest is therefore

$$g \sim x^{1/k} \left[1 - \frac{2k+1}{24k} x^{-(2k+1)/k} + \cdots \right] .$$

For the simplest case ($k=1$), g satisfies

$$g'' - \frac{g'^2}{2g} = 4g^2 - 4xg .$$

This equation is integrable in terms of known transcendents.[12] The equation satisfied by g in the case of $k=2$ is

$$6g^3 - \frac{15g'^4}{16g^3} - 5gg'' + \frac{9g'^2 g''}{4g^2} - \frac{3g''^2}{4g} - \frac{g' g^{(3)}}{g} + \frac{g^{(4)}}{2} - 6gx = 0 ,$$

which appears to be quite different from the fourth-order equations given in Refs. 2 and 8. Perhaps a deeper analysis of the underlying infinite-order equation will lead to a clear relation (or the lack of one) between the two sets of theories. In particular, this equation should be compared to Painlevé I for the Hermitian models. While the function f exhibits simple poles, the function g naturally has double poles, which is also the case for the scaling functions found in Refs. 2-4. Another similarity between these theories is the occurrence of additional parameters that are not evident in the perturbative expansion. We find that there are k such parameters, even though *a priori* there are $2k$ initial conditions that need to be specified for the differential equations. k of them are determined by matching the nonperturbative solution to the planar approximation. It is interesting that there is one additional nonperturbative parameter in the unitary models, as compared to the Hermitian models. Since the general planar theory has not been solved before, we give below the solution of the general $k=4$ model in the planar limit.

It is, we believe, of some interest to understand what conformal field theories are obtained from these unitary-matrix models. We think it is significant that the specific heat in these unitary-matrix models is positive definite. Quite obviously, this does not guarantee that the coefficients of a perturbative expansion of $g(t)$ in powers of the string constant will have any positivity properties.

Correlation functions in these models may be obtained[10] as simply as in the Hermitian-matrix models.[4] The properties of the correlations will be helpful in identifying the conformal field theories corresponding to these models. The relevant planar correlations may be derived easily from the formulas we give below.

We end this Letter by giving the general solution of the $k=4$ potential at arbitrary values of the coupling constants. It is then easy to see how the critical exponents change as these parameters are tuned. We shall be brief here because the essential analysis is contained in Refs. 5 and 6. The planar approximation amounts to approximating the functional integral by a saddle point. The basic idea[5] is to derive the distribution of the eigenvalues of the "dominant" unitary matrices, which is dictated by the form of v. In the large-N limit, the number of eigenvalues is infinite and, since they are all constrained to lie on a compact set (the unit circle), we introduce the density of eigenvalues $\rho(\alpha) \equiv dy/d\alpha \geq 0$, where y is a continuous variable going from 0 to 1, which corresponds to i/N for finite values of N. Then the saddle-point equation for determining ρ is

$$\frac{2}{\lambda} \sin(\alpha) v'(\cos\alpha) = P \int_{-\alpha_c}^{\alpha_c} \rho(\beta) \cot\left[\frac{\alpha-\beta}{2}\right] ,$$

with P denoting the principal part of the integral. (We have changed conventions here to those of Ref. 6, to facilitate comparison. The eigenvalues are now taken to run from $[-\pi,\pi]$, so that at *positive* small λ the eigenvalues are close to 0 in the simplest model. In general, one can pick the sign of the coupling so that this statement is valid at weak coupling.) α_c may take any value up to π, and is determined along with ρ by solving the equation above. The solution is obtained exactly as in Ref. 6. For

$$v(\cos(\alpha)) = \frac{2}{\lambda} \left[\cos(\alpha) + \frac{k_1}{2} \cos^2(\alpha) + \frac{k_2}{3} \cos^3(\alpha) + \frac{k_3}{4} \cos^4(\alpha) \right] ,$$

we find that the density of eigenvalues is given by

$$\rho(\alpha) = (2/\pi\lambda) \cos(\alpha/2) [a - \sin^2(\alpha/2)]^{1/2} \{ 1 - a(k_1+k_2+k_3) + 3a^2(k_2/2+k_3) - 5a^3(k_3/2) + [k_1 - a(k_2+k_3) + \tfrac{3}{2} a^2 k_3] \cos(\alpha) + (k_2 - ak_3)\cos^2(\alpha) + k_3 \cos^3(\alpha) \} ,$$

where $a \equiv \sin^2(\alpha_c/2)$. a satisfies

$$\lambda/2=(1+k_1+k_2+k_3)a-\tfrac{3}{2}(1+k_1+2k_2+3k_3)a^2+\tfrac{5}{2}(k_2+3k_3)a^3-\tfrac{35}{8}k_3a^4 .$$

The solution chosen is real, positive, and less than 1. There is a critical value of λ at which $a=1$. Upon tuning k_i, the exponent that characterizes the vanishing of the spectral density at criticality changes discontinuously, as expected exhibiting universal behavior.

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[1]V. A. Kazakov, Niels Bohr Institute Report No. NBI-HE-89-25, 1989 (to be published).

[2]E. Brézin and V. A. Kazakov, Ecole Normale Supérieure report, 1989 (to be published).

[3]M. Douglas and S. H. Shenker, Rutgers University Report No. RU-89-34, 1989 (to be published).

[4]D. J. Gross and A. A. Migdal, Princeton University Report No. PUPT-1148, 1989 (to be published).

[5]E. Brézin, C. Itzykson, G. Parisi, and J.-B. Zuber, Commun. Math. Phys. **59**, 35 (1978).

[6]D. J. Gross and E. Witten, Phys. Rev. D **21**, 446 (1980).

[7]M. Staudacher, University of Illinois Report No. ILL-(TH)-89-54, 1989 (to be published).

[8]D. J. Gross and A. A. Migdal, Princeton University Report No. PUPT-1156, 1989 (to be published).

[9]J. Cardy and J. Cohn (private communication) inform us that E. Brézin, M. Douglas, V. Kazakov, and S. Shenker have obtained results in this direction as well.

[10]V. Periwal and D. Shevitz (to be published).

[11]D. Bessis, Commun. Math. Phys. **69**, 147 (1979); C. Itzykson and J.-B. Zuber, J. Math. Phys. **21**, 411 (1980).

[12]E. L. Ince, *Ordinary Differential Equations* (Dover, New York, 1956).