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Boson-Vortex-Skyrmion Duality, Spin-Singlet Fractional Quantum Hall Effect, and Spin-$\frac{1}{2}$ Anyon Superconductivity

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In two space dimensions we generalize the boson-vortex duality picture for spinless particles to boson-vortex-Skyrmion duality for spin-$\frac{1}{2}$ particles. The spin-singlet fractional quantum Hall effect and spin-singlet anyon superconductivity can be understood as the condensation of vortices and Skyrmions into various fractional quantum Hall states.

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Important progress in the theory of the fractional quantum Hall effect (FQHE) was made by Girvin and McDonald and Read[1] when they identified a novel kind of long-range order (to be more precise, quasi-long-range order in the work of Girvin and McDonald and long-range order in the work of Read) hidden in the $\nu = 1/m$ Laughlin wave function. This long-range order arises from a "Bose condensation" of composite objects which are made of electrons bound to vortices in the many-particle wave function. (In this paper we use the term Bose condensation loosely in that we do not distinguish between true long-range order and power-law long-range order.) With suitable generalization, similar ideas can be used to characterize the hierarchical fractional quantum Hall states as well as the superconducting states of the anyon problem.[2] The discovery of the spin-singlet even-denominator fractional quantum Hall state[3] adds an additional dynamical degree of freedom, the spin of electrons, to the problem. The interesting question is how does the spin degree of freedom alter the so-called odd-denominator rule for the spin-polarized situation.

In this paper, we generalize the duality transformation[4] for the spinless FQHE and anyon superconductivity[5] to the spin-$\frac{1}{2}$ case. Heuristically, the duality may be understood as follows. First, let us concentrate on the spinless (spin-polarized) fermion FQHE. In two dimensions, a spinless electron may be represented as a hard-core boson carrying an odd integer of Dirac flux quanta.[6] The effect of the flux tubes is to induce an appropriate Aharonov-Bohm phase factor which simulates the statistical phase factor when two particles are adiabatically exchanged. If on average the statistical-flux density cancels the external magnetic-flux density, the bosons condense and acquire long-range phase coherence. Thus the $\nu = 1/m$ (m being an odd integer) FQHE can be viewed as the condensation of the hard-core bosons after their statistical flux (m Dirac flux quanta per boson) cancels the external flux. The order parameter of Girvin and McDonald is exactly the superfluid order parameter for these hard-core bosons. Since a hard-core boson can in turn be viewed as a fermion bound to an odd integer of Dirac flux quanta, an alternative view of the $\nu = 1/m$ fractional quantum Hall liquid is that after the external magnetic field induces on average m vortices per particle, each electron simply "swallows m vortices" to form a composite boson and Bose condense. If $\nu \neq 1/m$, there will be a residual density of vortices in the Bose field. Under duality,[4] which interchanges the roles of charges and vortices, these vortices may be regarded as Bose particles, which see each original boson as a flux tube carrying one Dirac flux quantum. Each of these new bosons may then bind with an even integer of the new vortices and Bose condense. By repeated applications of this duality transformation, the hierarchy of the FQHE may be generated.[5,7] In the same heuristic picture the semion superconducting state[5,8] can be viewed as follows. A semion is viewed as a hard-core boson carrying half of a Dirac flux quantum. Since there is no external magnetic

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field to cancel the statistical flux in the anyon problem, this statistical flux induces a density of vortices in the Bose field which corresponds to half of a vortex per boson. Like in FQHE, these vortices turn into Bose particles under duality and see each hard-core boson as a flux tube carrying one Dirac flux quantum. Since now the ratio between the number of new vortices and the number of new bosons is precisely 2 to 1, each new boson swallows two vortices and Bose condenses.

The spin degree of freedom introduces an additional topological excitation—the Skyrmion.[9] Inside a Skyrmion the spins wrap around the unit sphere. A finite Skyrmion appears like a flux tube carrying one Dirac flux quantum, a length scale large compared to its size; i.e., the Berry's phase one obtains by adiabatically moving a spin-$\frac{1}{2}$ particle around it is 2π.[10] In this case the external magnetic field and the statistical-flux tubes induce vortices and/or Skyrmions. Under the duality, we regard the vortices and the Skyrmions as independent Bose particles, which see the original particles as flux tubes. These particles may then condense into their own FQHE states, which generates a new hierarchy of filling fractions for the non-spin-polarized FQHE. We will argue that the situation with one-half of a Skyrmion per particle corresponds to a singlet state.

We begin by considering a model of spin-$\frac{1}{2}$ particles with any statistics, described by the following effective Lagrangian[11] ($\hbar = e = c = 1$):

$$\mathcal{L} = \bar{\Psi}_\sigma(\partial_0 - ia_0)\Psi_\sigma + \frac{1}{2m}\left|\left(\frac{\boldsymbol{\partial}}{i} - \mathbf{a} - \mathbf{A}_{\rm ex}\right)\Psi_\sigma\right|^2 + \tfrac{1}{2}u(\bar{\Psi}_\sigma\Psi_\sigma - \bar{\rho})^2 + \mathcal{L}_{\rm CS}, \tag{1}$$

$$\mathcal{L}_{\rm CS} = (i/4\pi\alpha)\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda .$$

Here and throughout the rest of the paper, repeated greek indices imply summation. Ψ_σ is a two-component Bose field with effective mass m and short-range repulsive interaction u, which couples to the external and statistical gauge fields $A_{{\rm ex},\mu}$, a_μ ($\mu = 0,1,2$); α is an even integer for bosons, an odd integer for fermions, and a fraction for anyons. The effect of the Chern-Simons term is to attach α flux quanta in a_μ to each boson. In order to separate the charge and spin degree of freedom, we explicitly separate the magnitude and the U(1) and SU(2) phases of Ψ_σ: $\Psi_\sigma \to J_0^{1/2}\phi z_\sigma$ with $\bar{\phi}\phi = \bar{z}_\sigma z_\sigma = 1$. By direct substitution and keeping the leading-order gradient terms we obtain

$$\mathcal{L} = \frac{u}{2}(J_0 - \bar{\rho})^2 + iJ_0\left(\bar{\phi}\frac{\partial_0}{i}\phi + \bar{z}_\sigma\frac{\partial_0}{i}z_\sigma - a_0\right) + \frac{K}{2}\left|\bar{\phi}\frac{\boldsymbol{\partial}}{i}\phi + \bar{z}_\sigma\frac{\boldsymbol{\partial}}{i}z_\sigma - \mathbf{A}_{\rm ex} - \mathbf{a}\right|^2 + \mathcal{L}_z + \mathcal{L}_{\rm CS}, \tag{2a}$$

where $K = \bar{\rho}/m$ and

$$\mathcal{L}_z = \tfrac{1}{2}K[|\boldsymbol{\partial} z_\sigma|^2 + (\bar{z}_\sigma\boldsymbol{\partial} z_\sigma)^2] = \tfrac{1}{8}K|\boldsymbol{\partial}\hat{\mathbf{n}}|^2. \tag{2b}$$

Here $\hat{\mathbf{n}} = \bar{z}_\alpha\boldsymbol{\sigma}_{\alpha\beta}z_\beta$ ($\boldsymbol{\sigma}$ are the Pauli matrices) is the spin direction corresponding to z_σ. We then introduce Hubbard-Stratonovich fields $\mathbf{J}$ to decouple the third term of (2a) to obtain

$$\mathcal{L} = \frac{u}{2}(J_0 - \bar{\rho})^2 + \frac{1}{2K}|\mathbf{J}|^2 + iJ_\mu\left(\bar{\phi}\frac{\partial_\mu}{i}\phi + \bar{z}_\sigma\frac{\partial_\mu}{i}z_\sigma\right) + i\mathbf{J}\cdot\mathbf{A}_{\rm ex} + iJ_\mu a_\mu + \mathcal{L}_z + \mathcal{L}_{\rm CS}, \tag{3}$$

where $J_\mu = (J_0, \mathbf{J})$. After integrating out the longitudinal fluctuations in ϕ, we obtain $\partial_\mu J_\mu = 0$, so J_μ is a conserved current. This conservation constraint is then explicitly satisfied by writing $J_\mu = (1/2\pi)\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda$. In view of the space-time anisotropy we then choose the Coulomb gauge $\boldsymbol{\partial}\cdot\mathbf{A} = 0$, $\boldsymbol{\partial}\cdot\mathbf{a} = 0$ and integrate out A_0 and a_0 to obtain

$$\mathcal{L} = \frac{u}{8\pi^2}(\boldsymbol{\partial}\times\mathbf{A} - 2\pi\bar{\rho})^2 + \frac{1}{8\pi^2K}|\partial_0\mathbf{A}|^2 + \mathcal{L}_z + \pi K\mathcal{L}_I\left(J_0^v + J_0^S + \frac{\alpha}{2\pi}\boldsymbol{\partial}\times\mathbf{A} - H_0\right) + \frac{1}{2K_v}|\mathbf{J}^v|^2 + \frac{1}{2K_S}|\mathbf{J}^S|^2 + i\mathbf{A}\cdot(\mathbf{J}^v + \mathbf{J}^S). \tag{4}$$

Here

$$J_\mu^v = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu\bar{\phi}\frac{\partial_\lambda}{i}\phi,$$

$$J_\mu^S = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu\bar{z}_\sigma\frac{\partial_\lambda}{i}z_\sigma = \frac{1}{4\pi}\epsilon_{\mu\nu\lambda}(\partial_\nu\hat{\mathbf{n}}\times\partial_\lambda\hat{\mathbf{n}})\cdot\hat{\mathbf{n}}$$

are the vortex and Skyrmion[12] three-currents, respectively. Moreover, $\mathcal{L}_I[\rho(r)] = \int dr'\rho(r)\ln|r - r'|\rho(r')$ and $H_0 = (1/2\pi)\boldsymbol{\partial}\times\mathbf{A}_{\rm ex}$ is the external magnetic field measured in units of the Dirac flux quantum. It is important to realize that the flux of the new gauge field $\mathbf{A}$ is the density of the original bosons. The vortex and Skyrmion hopping terms $(1/2K_v)|\mathbf{J}^v|^2$ and $(1/2K_S)|\mathbf{J}^S|^2$ are added to reflect the effects of short-wavelength and high-frequency cutoff. Since $\int d^2r\,\mathcal{L}_I[\rho(r)]$ diverges if $\int d^2r \times\rho(r) \neq 0$, the combination of magnetic and statistical flux will induce Skyrmions and/or vortices. Moreover, as indicated by the $\mathbf{A}\cdot\mathbf{J}^v$ and $\mathbf{A}\cdot\mathbf{J}^S$ terms, when Skyrmions and vortices move, they see the original particles as magnetic fluxes.

To obtain an effective theory for vortices and Skyrmions, we integrate out the remaining SU(2) degrees of freedom:

$$\int D[z_\sigma]\exp\left(-\int d\tau d^2r\,\mathcal{L}_z\right)\delta\left(J_\mu^S - \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu\bar{z}_\sigma\frac{\partial_\lambda}{i}z_\sigma\right) = \exp\left(-\int d\tau d^2r\,\mathcal{L}_{\rm eff}(J_\mu^S)\right). \tag{5}$$

While we have no explicit way of doing this, given the fact that $\mathcal{L}_z$ is a two-dimensional nonlinear σ model with no dynamics and the 2D nonlinear σ model is disordered at any finite coupling constant, we expect $\mathcal{L}_{\rm eff}$ will just introduce an instantaneous interaction between the Skyrmions and some correction to the Skyrmion mass.

In view of the long-range logarithmic interaction between the Skyrmions in (4), we believe that $\mathcal{L}_{\text{eff}}$ is irrelevant at long wavelengths and low energies. Strictly speaking, the effective vortex-Skyrmion Lagrangian we obtain after performing (5) does not contain the full dynamical information. It is the dual theory of (1) in the sense that all low-energy spin excitations correspond to the motions of Skyrmions. Given (4), we expect that at particular densities, corresponding to the boson FQHE filling factors, vortices and Skyrmions may condense into their own boson FQHE states. In such a case the ground state will in general be an incompressible liquid.

We will now consider some specific applications of (4). We first consider a case where we already know the answer: the $\nu=2$ quantum Hall effect. In that case, there are two possible incompressible states. In the first case the spins of the electrons can all be polarized to fill two Landau levels. Alternatively, it is also possible to have a singlet state corresponding to spin-up and spin-down electrons filling a single Landau level.

In order to describe hard-core fermions, we attach one statistical-flux quantum to each hard-core boson, i.e., $\alpha=1$. In this case the sum of the statistical and external-flux density gives an average field strength of $\frac{1}{2}$ flux quanta per particle. Through the $\mathcal{L}_I$ term in (4), this residual flux density will induce a vortex or a Skyrmion per every two bosons. Two filled spin-polarized Landau levels correspond to the case where the vortices condense into the $\nu=\frac{1}{2}$ FQHE state; hence the condensate boson consists of a vortex bound to two spin-polarized particles. By performing the duality transformation once more for the vortices, we obtain the following effective Lagrangian describing the collective excitations around this incompressible state:

$$\mathcal{L}=\frac{u}{8\pi^2}(\boldsymbol{\partial}\times\mathbf{A}-2\pi\bar{\rho})^2+\frac{1}{8\pi^2K}|\partial_0\mathbf{A}|^2+\frac{1}{8\pi^2K_v}|\partial_0\mathbf{A}_v|^2+\frac{K}{4\pi}\mathcal{L}_I[\boldsymbol{\partial}\times\mathbf{A}-\boldsymbol{\partial}\times\mathbf{A}_v-2\pi H_0]+\frac{K_v}{4\pi}\mathcal{L}_I[2\boldsymbol{\partial}\times\mathbf{A}_v-\boldsymbol{\partial}\times\mathbf{A}]. \tag{6}$$

By diagonalizing the quadratic Lagrangian we obtain two normal modes whose dispersions at long wavelength are $\omega(\mathbf{q}\to 0)=2\pi\omega_\pm$, where

$$\omega_\pm^2=[(K^2+4K_v^2+2KK_v)\pm(K+2K_v)(K^2+4K_v^2)^{1/2}]/2.$$

Therefore both normal modes are massive, so there is a gap for long-wavelength density fluctuations.

We may alternatively condense Skyrmions, rather than vortices. In this case, everything follows in exactly the same fashion as above, except that now the condensed object consists of two particles and a Skyrmion. The effective Lagrangian for collective excitations and the corresponding dispersions can be easily obtained from that of (6) and $\omega_\pm$ by replacing $K_v\to K_S$ and $\mathbf{A}_v\to\mathbf{A}_S$. For both cases we have computed the longitudinal and transverse conductivity and obtain $\sigma_{xx}(\omega\to 0)\to 0$ and $2\pi\sigma_{xy}(\omega=0)=2$.

We now argue that the state we obtain in the second case is the spin-singlet singly filled Landau level. To do so let us compare the nature of the quasihole we deduced from the effective Lagrangian approach to that obtained from the wave functions. The wave function corresponding to the $\nu=2$ singlet filled Landau level is

$$\Psi(\{z\},\{\eta\})=\prod_{i,j}(z_i-z_j)\prod_{k,l}(\eta_k-\eta_l)\exp\left[-\tfrac{1}{4}\sum(|z_i|^2+|\eta_k|^2)\right], \tag{7a}$$

where z and η are the complex coordinates of the spin-up and spin-down electrons, respectively. The quasihole wave functions are given by

$$\Psi_{z_0}=\prod_i(z_i-z_0)\Psi,\quad \Psi_{\eta_0}=\prod_i(\eta_i-\eta_0)\Psi, \tag{7b}$$

where z_0 and η_0 are the complex coordinates of the quasiholes. These quasihole wave functions describe a spin-$\frac{1}{2}$ (spin-up or -down) charge $+e$ fermion excitation. In particular, in a quasihole described by Ψ_{z_0} there is a local depletion of one spin-up particle around z_0. Therefore the interior of a quasihole is a ferromagnetic bubble with averaged spin pointing down. Now, let us see what the effective Lagrangian approach has to say about the quasiholes. To create a quasihole we force in a vortex in the Skyrmion condensate; i.e., in the last term of (6) we replace $2\boldsymbol{\partial}\times\mathbf{A}_S-\boldsymbol{\partial}\times\mathbf{A}$ by $J_0^{Sv}+2\boldsymbol{\partial}\times\mathbf{A}_S-\boldsymbol{\partial}\times\mathbf{A}$, where $J_0^{Sv}(r)=\delta^2(r-r_0)$ is the vortex density in the Skyrmion field. To cancel the cost in the logarithmically divergent energy $\mathbf{A}$ and $\mathbf{A}_S$ have to relax such that

$$\frac{1}{2\pi}\int d^2r[2\boldsymbol{\partial}\times\mathbf{A}_S-\boldsymbol{\partial}\times\mathbf{A}]=-1,$$

$$\int d^2r[\boldsymbol{\partial}\times\mathbf{A}-\boldsymbol{\partial}\times\mathbf{A}_S]=0,$$

where the integral extends over the area on the order of magnetic length squared. Therefore both the integrated particle-number deficit and the integrated Skyrmion-number deficit are 1. Because the quasihole is a bound state of a bosonic charge and a Skyrmion, it has the statistics of a fermion. Comparing the results of the effective Lagrangian approach and the wave-function approach, we therefore draw the connection that the local ferromagnetic bubble is the remnant of the local depletion of one Skyrmion. Moreover, since the Skyrmion density fluctuation is massive, we expect the ferromagnetic bubble to have a finite size. Since the ferromagnetic bubble is the remnant of the removal of a Skyrmion we conclude that the Skyrmions have a finite size (from the wave-function approach we can deduce this size to be on the order of the magnetic length). This correspondence between a local unpaired spin and the removal of one Skyrmion is completely consistent with the interpretation that the singlet state (7a) (which is made up of $N/2$ spin-up and $N/2$ spin-down electrons) corresponds to the condensation of $N/2$ Skyrmions.[13] Despite the consistency of the effective Lagrangian approach and the wave-function approach, we cannot prove rigorously that condensation of $\frac{1}{2}$ Skyrmion per particle produces a spin-singlet state.

We now consider the $\nu=\frac{1}{2}$ singlet FQHE. We put $\alpha=1$ in order to represent hard-core fermions as hard-core bosons carrying one Dirac flux quantum. The $\mathcal{L}_I$ term in (4) induces an average density of vortices and Skyrmions such that $J_0^v+J_0^S=\bar{\rho}$. It is possible to construct a condensed state of vortices and Skyrmions if $J_0^v=J_0^S=\bar{\rho}/2$. In that case both the vortices and the Skyrmions condense into the $\nu=\frac{1}{2}$ FQHE state and the effective Lagrangian for the collective excitations is given by

$$\mathcal{L}=\frac{u}{8\pi^2}(\boldsymbol{\partial}\times\mathbf{A}-2\pi\bar{\rho})^2+\frac{1}{8\pi^2K}|\partial_0\mathbf{A}|^2+\frac{1}{8\pi^2K_v}|\partial_0\mathbf{A}_v|^2+\frac{1}{8\pi^2K_S}|\partial_0\mathbf{A}_S|^2$$
$$+\frac{K}{4\pi}\mathcal{L}_I[\boldsymbol{\partial}\times\mathbf{A}_v+\boldsymbol{\partial}\times\mathbf{A}_S+\boldsymbol{\partial}\times\mathbf{A}-2\pi H_0]+\frac{K_v}{4\pi}\mathcal{L}_I[2\boldsymbol{\partial}\times\mathbf{A}_v-\boldsymbol{\partial}\times\mathbf{A}]+\frac{K_S}{4\pi}\mathcal{L}_I[2\boldsymbol{\partial}\times\mathbf{A}_S-\boldsymbol{\partial}\times\mathbf{A}]. \quad (8)$$

To obtain the collective-mode dispersion one has to diagonalize the 3×3 secular matrix of the Lagrangian. Because of the presence of the external-flux density, all three normal modes are in general massive at $q=0$. We have also calculated $\sigma_{xx}(\omega)$ and $\sigma_{xy}(\omega)$ and the results are just as expected, i.e., $\sigma_{xx}(\omega\rightarrow0)\rightarrow0$ and $2\pi\sigma_{xy}(\omega=0)=\frac{1}{2}$. In general, we may continue further into the hierarchy by iterating the duality. If we consider higher hierarchies in the Skyrmions, then we will in general have different values of the total spin (since $J_0^S=\bar{\rho}/2$ corresponds to a singlet state and $J_0^S=0$ corresponds to the spin-polarized state). If we restrict ourselves to $J_0^S=\bar{\rho}/2$, i.e., the singlet state, then we may generate the hierarchy for fermionic spin-singlet FQHE:

$$\nu_{S=0}=1/(m+\tfrac{1}{2}+\nu_b), \quad (9)$$

where m is an odd integer and ν_b are the hierarchical filling fractions of the spinless boson FQHE. Some of the low-order fractions in this hierarchy are $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{4}{7}$, $\frac{4}{11}$.

Finally, we address the question of superconductivity for spin-$\frac{1}{2}$ semions.[14] In this case, we set $\alpha=\frac{1}{2}$ in order to represent semions in terms of hard-core bosons. In this case the $\mathcal{L}_I$ term in (4) will induce an average density $\bar{\rho}/2$ of either Skyrmions or vortices. If it induces vortices, then they see an average field corresponding to $\nu=\frac{1}{2}$, so that they will condense. This corresponds to a triplet pairing of the semions. Alternatively, Skyrmions may be induced. In that case, the Skyrmions will condense into the $\nu=\frac{1}{2}$ FQHE state, which corresponds to a singlet superconducting state of the semions. The effective Lagrangian for collective excitations is

$$\mathcal{L}=\frac{u}{8\pi^2}(\boldsymbol{\partial}\times\mathbf{A}-2\pi\bar{\rho})^2+\frac{1}{8\pi^2K}|\partial_0\mathbf{A}|^2+\frac{1}{8\pi^2K_S}|\partial_0\mathbf{A}_S|^2+\frac{K}{4\pi}\mathcal{L}_I[\boldsymbol{\partial}\times\mathbf{A}_S-\tfrac{1}{2}\boldsymbol{\partial}\times\mathbf{A}]+\frac{K_S}{4\pi}\mathcal{L}_I[2\boldsymbol{\partial}\times\mathbf{A}_S-\boldsymbol{\partial}\times\mathbf{A}]. \quad (10)$$

The collective-mode dispersions are

$$\omega_1(\mathbf{q})=\left(\frac{4uKK_S}{K+4K_S}\right)^{1/2}|\mathbf{q}|,\quad \omega_2(\mathbf{q})=\pi(K+4K_S).$$

The longitudinal and transverse conductivities are

$$2\pi\sigma_{xx}(\omega\rightarrow0)\rightarrow\frac{2\pi}{i\omega}\frac{4KK_S}{K+4K_S}$$

and

$$2\pi\sigma_{xy}(\omega=0)=2\left(\frac{K}{K+4K_S}\right)^2.$$

The vortex of the semion superconducting state can be created by replacing $2\boldsymbol{\partial}\times\mathbf{A}_S-\boldsymbol{\partial}\times\mathbf{A}$ by $J_0^{Sv}+2\boldsymbol{\partial}\times\mathbf{A}_S-\boldsymbol{\partial}\times\mathbf{A}$, where J_0^{Sv} is the vortex density. By integrating out $\mathbf{A}$ and $\mathbf{A}_S$ we obtain the effective interaction between the vortices,

$$V[J_0^{Sv}]=\pi\frac{KK_S}{K+4K_S}\int d^2r\,d^2r'\,J_0^{Sv}(r)\ln|r-r'|J_0^{Sv}(r').$$

Because of the logarithmic interaction between the vortices, singlet semion superconductivity survives nonzero temperature. The phase transition will presumably be triggered by vortex unbinding similar to the U(1) semion problem.

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