Nonperturbative Two-Dimensional Quantum Gravity

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We propose a nonperturbative definition of two-dimensional quantum gravity, based on a double-scaling limit of the random-matrix model. We derive an exact differential equation for the partition function of two-dimensional gravity coupled to conformal matter as a function of the string coupling constant that governs the genus expansion of two-dimensional surfaces, and discuss its properties and consequences. We also construct and discuss the correlation functions of an infinite set of local operators for spherical topology.

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The conventional approach to two-dimensional gravity and to string theory[1] is perturbative with respect to fluctuations of the topology. One sums over two-dimensional geometries by first performing the functional integral for fixed topology (fixed genus equal to the number of handles) and then summing over genus. However, this sum is very badly behaved. The higher terms grow as factorials of the genus, and the positivity of these terms renders the series non-Borel summable.[2] It would appear that we are faced with a genuine nonperturbative phenomenon, like quark confinement, made worse by the absence of a nonperturbative definition of the theory. Such a framework (for example, a useful formulation of second-quantized string theory) should be capable of reproducing the topological series as an asymptotic expansion, valid in the perturbative domain; but it should also provide a physical picture and a mathematical framework valid for strong coupling. From what we already know about gravity and strings, we expect dramatic phenomena in this region.

Recently, a completely different approach to gravity and string theories has been pursued.[3] The geometry of the world sheet of the string (or two-dimensional space in the case of pure gravity) is approximated by a dense Feynman graph of the same topology, in the limit where the number of vertices becomes infinite. The topology is selected by means of the $1/N$ expansion of a SU_N-invariant Hermitian-matrix model, while an infinite number of vertices can be produced by adjusting the coupling constant to equal a critical value at which the loop expansion of the matrix model begins to diverge. The sum over all Feynman graphs of given genus and given number of vertices can be regarded as a discrete version of the functional integral over metric tensors. Remarkably, in many cases these discrete models can be handled with greater ease than their continuous analogs. Much work has been done for dimensions less than one and spherical topology, with results that are in complete agreement with those of conformal field theory.[4,5] [There have also been some interesting observations[6] concerning higher-genus surfaces in the supersymmetric case ($d=-2$), which is particularly simple since the random matrix can be expressed in terms of a free field (i.e., a Gaussian matrix).]

The matrix approach has numerous advantages. It allows for efficient computer simulations and it makes possible powerful combinatorial methods that often enable one to explicitly solve the discrete models. This is a great advantage over continuum methods, which rarely allow for exact solution of model with an ultraviolet cutoff. Finally, it makes sense beyond the $1/N$ expansion. It is the last point that we shall develop here.

We regard the partition function of the random Hermitian-matrix model,

$$Z_N(\beta)=\int d\Phi \exp[-\beta \operatorname{Tr} U(\Phi)]\,, \tag{1}$$

as the discrete version of the sum over surfaces $\mathcal{S}$, of genus G and area A. The logarithm of the original partition function,

$$\ln Z_N(\beta)=\text{regular terms}+\sum_{\mathcal{S}} N^{2(1-G)}\left(\frac{N}{\beta}\right)^A A^{-1}F_{\mathcal{S}}[U]\,, \tag{2}$$

generates connected graphs. The irrelevant regular terms arise when the matrix field Φ is rescaled to renormalize to $\Phi^2/2$ the quadratic term in $\beta U(\Phi)$. The role of the area of the graph is played by the number of loops of the dual graph. The factor A^{-1} compensates for the overcounting of loops in the evaluation of the vacuum energy (this factor would disappear in the calculation of correlation functions). The factor $F_{\mathcal{S}}[U]$ is given by the sum of the products of the vertex weights corresponding to the cubic and higher-order terms in $U(\Phi)$, divided by the order of the symmetry group of the graph with one marked loop. (The factor A^{-1} takes care of the A ways to mark the loop.) This sum runs over all graphs with the same number of loops and the same genus. The continuum limit is achieved by carefully adjusting β so that the loop expansion diverges.

Recently Kazakov[7] made a remarkable observation. He noted that, by adjusting the parameters U_k, one can reproduce the critical behavior of matter coupled to gravity; i.e., a carefully constructed $F_{\mathcal{S}}[U]$ can yield the partition function of conformally invariant matter fields in a gravitational background. (In order to obtain the multicritical points one must have negative weights for some triangulations. However, as we shall see, physical positivity can be preserved.) Kazakov has explicitly verified this conjecture for spherical topology.[7] As we shall see below, the universality of the critical behavior holds to all orders of the $1/N$ expansion, but the nonperturbative terms introduce $k-1$ extra parameters. (See also the interesting paper by Douglas and Shenker,[8] where many of our results were independently obtained. They were the first to realize the significance of the higher-derivative terms for $k>2$, which we inadvertently doped in an earlier version of this Letter.)

To be specific, we shall take the double limit: $N\to\infty$, $\beta/N\to 1$, and adjust k parameters in the potential $U(\Phi)$, after which the following scaling law will hold:

$$\ln Z_N(\beta)=\text{regular terms}-F(t),\qquad t\equiv(\beta-N)\beta^{-1/(2k+1)}. \tag{3}$$

We shall prove that the specific heat, $f(t)=\ddot{F}(t)$, obeys the following differential equation, which we propose as the basis of a nonperturbative definition of two-dimensional quantum gravity:

$$t=\frac{k!}{(2k-1)!!}\hat{K}[f(t),\nabla_t]^k\cdot 1,\qquad \hat{K}[f(t),\nabla_t]\equiv-\tfrac{1}{2}\nabla_t^2+f(t)+\nabla_t^{-1}f(t)\nabla_t. \tag{4}$$

The operator $\hat{K}$ was introduced by Gelfand and Dikii in their study of higher-order Korteweg-de Vries equations.[9] The nonlocal terms involving ∇_t^{-1} in this expression cancel. In fact, $\hat{K}^l\cdot 1$ yields the higher Korteweg–de Vries equations, $\hat{K}\cdot 1=f$, $\hat{K}^2\cdot 1=\frac{1}{2}(3f^2-\ddot{f})$, etc.

Equation (4) is universal, depending only on the single parameter k. The simplest case, $k=2$, corresponding to pure gravity, yields the Painlevé equation, $t=f^2-\frac{1}{3}\ddot{f}$, while the case of general k, which according to Kazakov[7] corresponds to gravity coupled to conformal matter with central charge given by $C=1-6/k(k+1)$, yields a $(2k-2)$th-order differential equation. The above-mentioned violation of perturbative universality corresponds to the ambiguity in the Cauchy data for this ordinary differential equation. We find that half of the free parameters are fixed by requiring that the asymptotic behavior at infinity correspond to spherical topology, i.e., $f\to t^{1/k}$. The remaining $k-1$ free parameters violate perturbative universality. We do not know of any general argument to fix them.

Let us now briefly describe the technique and the basic results of our approach to quantum gravity (full details are in Ref. 10). The first step is standard—we eliminate the angular matrices from (1), obtaining an integral over the eigenvalues ϕ_i of the matrix Φ,

$$Z_N(\beta)\propto\int\prod_{i=1}^{N}d\phi_i\,\Delta_N^2\exp\left[-\sum_{i=1}^{N}\beta U(\phi_i)\right],\qquad \Delta_N=\prod_{1\le i<j\le N}(\phi_j-\phi_i). \tag{5}$$

This is the partition function of a one-dimensional Coulomb gas of N equal charges in an external potential U, first introduced by Dyson.[11] Because of remarkable properties of one-dimensional Coulomb forces it can be exactly computed. The point is that the Coulomb factor Δ_N in the statistical weight coincides with the Vandermonde determinant $\Delta_N=\det\|\phi_i^{j-1}\|$, which enables us to apply the powerful theory of orthogonal polynomials.[12]

One introduces a space of functions, $F(\phi)$, with scalar product $\langle A\,|\,B\rangle\equiv\int d\phi\exp[-\beta U(\phi)]A(\phi)B(\phi)$. The basis vectors in this space, $|n\rangle$, correspond to orthogonal polynomials with weight $\exp[-\beta U(\phi)]$. Because of the orthogonality of the polynomials, the variable ϕ is represented in this basis by a tridiagonal operator, $\hat{\phi}|m\rangle=|m+1\rangle+R_m|m-1\rangle+S_m|m\rangle$. (We choose to normalize to unity the coefficient of the highest term in $|n\rangle$, so that $\langle n|n\rangle=R_n\langle n-1|n-1\rangle$ instead of unity, and ϕ is not manifestly Hermitean. This simplifies the intermediate equations.) These parameters R_m,S_m govern the recursion relation for the orthogonal polynomials and can be used to extract the physics from the model. Thus the partition function can be evaluated, for large N, as a product of R_n,[12]

$$Z_N(\beta)\propto\prod_{n=1}^{N-1}R_n^{N-n}\sim\exp\left[\beta^2\int_0^X dx(X-x)\ln R(x)\right], \tag{6}$$

where we introduce the continuous variables $x=n/\beta$, $X=N/\beta$, and $R_n\to R(x)$. R_n and S_n in turn satisfy the nonlinear recursion relations[10,12]

$$\frac{n}{\beta}\delta_{\sigma,1}\langle n-\sigma|n-\sigma\rangle=\langle n-\sigma|U'(\hat{\phi})|n\rangle,\quad \sigma=0,1, \tag{7}$$

which we shall use to calculate them. To this end it is very convenient to interpret $\hat{\phi}$ as an operator in the basis of eigenstates $|n\rangle$ of an angular momentum operator $\hat{l}$ [i.e., $\hat{l}|n\rangle=(n/\beta)|n\rangle$], conjugate to the angular coordinate θ; $\hat{\phi}=e^{i\theta}+e^{-i\theta}R(\hat{l})+S(\hat{l})$. It is then easy to see that (7) can be rewritten as

$$x\delta_{\sigma,1}=\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}e^{i\sigma\theta}U'\left[e^{i\theta}+e^{-i\theta}R\left(x-\frac{i}{\beta}\frac{d}{d\theta}\right)+S\left(x-\frac{i}{\beta}\frac{d}{d\theta}\right)\right]\cdot 1,\quad \sigma=0,1. \tag{8}$$

Let us first consider the limit $\beta\sim N\to\infty$, but $\beta/N>1$, which corresponds to spherical topology. In that case we drop the derivative terms in (8) and these equations can be interpreted as giving the extrema of

$$\Omega(R,S)\equiv-xS+\oint\frac{dz}{2\pi i}U\left[z+\frac{R}{z}+S\right]. \tag{9}$$

The critical behavior can be analyzed using the Landau theory of phase transitions. The generic singularity of the partition function is rational, $\ln Z_N(\beta)\sim$regular terms$+\beta^2(X-X_{\rm critical})^{1+p/q}$, in perfect agreement with the continuum result,[1] provided $C=1-6(p-q)^2/pq$.

From now on we restrict ourselves to the case of even potentials, where $S=0$. Equation (8) is simpler in terms of the following function $W(R)$:

$$W(R)=\oint\frac{dz}{2\pi i}U'\left(z+\frac{R}{z}\right),\qquad U(\phi)=\int_0^1\frac{dt}{t}W(t(1-t)\phi^2). \tag{10}$$

Indeed, (8) reduces to $x=W(R)$. The possible types of critical behavior can be deduced directly from this equation, after which one may reconstruct the potential $U(\phi)$ that produces this behavior, and then reinterpret the results from the point of view of the Dyson gas or in terms of random graphs. The scaling laws arise when $1-W(R)$ and $k-1$ of its derivatives vanish at, say, $R=1$; in other words, $W=W_k(R)=1-(1-R)^k$. (For even k this yields a potential that is unbounded from below, however, this does not affect the universal critical behavior, as we can cut off the potential for large ϕ and the net effect will be exponentially small terms that do not survive the scaling limit.[10]) The partition function in this limit behaves as

$$Z_N(\beta)\propto\exp\left(-\int_0^T dt(T-t)f(t)\right), \tag{11}$$

where

$$T=\beta-N/\beta^{1/(2k+1)},\quad f(t)=(1-R)\beta^{2/(2k+1)}.$$

Let us now return to the general scaling limit where we will sum the complete topological expansion using (4). We first note that the integral is dominated, for large β, by the region of small $\theta\sim1/\beta$. The following trick allows us to pick out the dominant terms. Define the singular potential $U_\nu(\phi)=B(\frac{1}{2},-\frac{1}{2}-\nu)(2-\phi)^{\nu+1/2}+(\phi\to-\phi)$. One may easily check that this potential gives $W=W_\nu(R)=(1-R)^\nu[1+o(1-R)]$, in the sense of analytic continuation from negative ν. For this potential (8) reduces to the form $x=-2\nu B(\frac{1}{2},\frac{1}{2}-\nu)\times\langle x|\hat{H}^{\nu-1/2}|x\rangle$, where H is the Schrödinger operator $H\equiv\hat\theta^2+1-R(\hat{t})$. Note that here $\hat{t}$ ($\hat\theta$) plays the role of coordinate (momenta). This function, especially in the limit of integer ν of interest to us, was studied by Gelfand and Dikii.[9] Using their results we obtain in the scaling limit the following closed expression [$u\equiv R(x)-R(x')$, $\nabla\equiv(1/\beta)d/dx'$]:

$$\begin{aligned}x&=\int_0^1\frac{dt}{2(1-t)^{1/2}}W\left[R(x)+\frac{t}{2(1-t)}\left(-\tfrac{1}{2}\nabla^2+u+\frac{1}{\nabla}u\nabla\right)\right]\cdot1\bigg|_{x'=x}\\&=W(R)+\frac{1}{6\beta^2}[W''(R)R''+\tfrac{1}{2}W'''(R)R'^2]+\frac{1}{60\beta^4}\{W'''R^{(4)}+W^{(4)}[\tfrac{3}{2}(R'')^2+2R'R''']\}\\&\quad+\cdots+\frac{2}{\beta^{2k-2}(2k-1)!!2^k}W^{(k)}R^{(2k-2)}+\cdots.\end{aligned} \tag{12}$$

Equation (12) is the basic dynamical equation of our theory. It is not too difficult, using (12) for $W(R)=1-(1-R)^k$, to show that the expansion terminates and that $f(t)$ obeys Eq. (4).

We may actually consider a more general potential, which corresponds to small perturbations (both relevant and irrelevant; since we can construct the theory explicitly before removing the cutoff we can construct irrelevant operators) of the kth multicritical point,

$$W(R)=W_k(R)-\sum_i\mu_i(W_{l_i}(R)-W_{l_i+1}(R))/\beta N. \tag{13}$$

The additional terms, when transformed to U by (10), represent the perturbation of the kth multicritical pont by the set of operators $O_i=(1/N)\mathrm{Tr}[U_{l_i}(\Phi)-U_{l_i+1}(\Phi)]$, with scaling dimensions $d_i=l_i/k$ (since as $1-W_k$ scales as $1-x$ and hence has dimension 1 for the kth multicritical model). These operators are the random-matrix counterparts of the Zamolodchikov multiscaling perturbations in conformal field theory. The coefficients μ_i in front of the operators have the meaning of chemical potentials or sources. The derivatives $\partial^n(\ln Z_N)/\partial\mu_1\cdots\partial\mu_n$ can be interpreted as connected correlation functions of these operators.

In order to determine the full genus dependence of the correlation functions it is necessary to solve a set of *linear* differential equations whose coefficients depend on $f(t)$ and its derivatives. However, on the sphere the problem is purely algebraic and can be completely solved using the Lagrange formula for series inversion. The result is amazingly simple (negative powers of derivatives stand for integrals),

$$\langle O_1\cdots O_n\rangle=N^{2(1-n)}k^{-1}(d/dy)^{n-3}y^{\Sigma-1},\qquad y=\frac{\beta}{N}-1,\quad \Sigma=\frac{1}{k}+\sum_{i=1}^n d_i. \tag{14}$$

A basis of orthogonal operators can be constructed (the two-point function given above defines a positive definite metric). (These results will be described at length in Ref. 10.) It would be extremely interesting to calculate similar correlation functions in the conventional path-integral approach and to compare with our results.

Let us, however, proceed with the main theme of this paper: The study of nonperturbative two-dimensional quantum gravity as described by Eq. (4). For $k=2$ this is the classical Painlevé I equation. Much is known about the solutions of the Painlevé equations; in particular, they have a one-parameter family of solutions which are finite for positive t and approach $\sqrt{t}$ at ∞. This is the boundary condition needed to reproduce the correct leading behavior in the perturbative limit. [Recall that $g_{\text{string}}^2=(1/t)^{2+1/k}$ is the string coupling constant.] The remaining free parameter λ can be regarded as the coefficient in front of the exponential correction to the asymptotic expansion. In the case of general k, we can derive, using (12), that

$$f(t) \to t^{1/k} - [(k-1)/12k]t^{-2} - \cdots - \sum_i \lambda_i t^{\nu_i} \exp\left[-\frac{4k}{2k+1}\xi_i^{(k)} t^{1+1/2k}\right] + \cdots . \quad (15)$$

[It is easy to prove, using (4), that the terms in the perturbative, large-t, expansion behave, in order n, as $(2n)!$ as expected. For details, see Ref. 10.] The ξ_i are the roots of the Gegenbauer polynomial, $C_{2k-1}^{\frac{1}{2}-k}(\xi)$, with positive real part, of which there are precisely $k-1$ in number. The $k-1$ roots with negative real part give growing exponentials that must be killed to give the correct large-t behavior. The coefficients of the exponentially decreasing terms, the λ_i, are the $k-1$ free parameters that cannot be seen in perturbation theory.

This exponential correction can be interpreted as a kind of gravitational instanton effect. The peculiar power of t that appears in the exponential is precisely of order $\beta \sim N$, which is the square root of the inverse of the topological (string) coupling constant. What kind of string field theory could yield this kind of instanton?

As we go deeper into the region of strong coupling (of small t), these exponential corrections grow and start interacting via the nonlinearities of the equation. Eventually this must lead to collapse, as we can see by the following argument. The interpretation of the partition function (2) as a sum over random surfaces requires, at the very least, that all derivatives of $\ln Z_N(\beta)$ with respect to $1/\beta$ be positive. This means that odd (even) derivatives of $f(t)$ must be positive (negative). Consider the simplest case of $k=2$. We have verified that this property holds, order by order in the perturbative expansion; however, the basic equation does not guarantee that this will persist to strong coupling. Indeed, from the Painlevé equation we deduce, assuming the above positivity condition, that the solution of (4) that is positive for $t \geq 0$ must satisfy $1/2\sqrt{t} < \dot{f} < (3t)^{1/2}$, which clearly cannot be satisfied for $t < 1/2\sqrt{3}$. The actual value of g_{string}^2 at which the collapse takes place depends on the free parameter λ. Note that this violation of positivity does not represent a singularity of the Dyson gas; only our assumptions (perhaps false) about the interpretation of quantum gravity as a sum over surfaces are violated. The issue of the possible existence of a strong-coupling phase of quantum gravity is of great importance, especially in string theory, where g_{string}^2 is a dynamical parameter. [We shall analyze (4) and this issue at length in a longer version of this Letter.[10]]

Can we pass beyond the singularity at $C=1$ to dimension greater than one? This is not trivial. As $C \to 1$ (and $k \to \infty$) our potential becomes infinitely steep; the order of the equation becomes infinite as does the number of apparently free parameters. The region $1 < C < 25$ would correspond to imaginary $2k+1$ which does not have any meaning as far as we can see. The region $C > 25$ corresponds to negative $2k+1$ which causes other problems. Thus, the nonperturbative solution of string theory in physical dimensions remains to be found. However, we believe that it is of great value to study in detail the nonperturbative properties of the theory for $C < 1$ as this will have much to teach us about future formulations of string theories.

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[1]A. Polyakov, Phys. Lett. **103B**, 207 (1981); Mod. Phys. Lett. A **2**, 899 (1987); V. Knizhnik, A. Polyakov, and Z. Zamolodchikov, Mod. Phys. Lett. A **3**, 819 (1988).

[2]D. Gross and V. Periwal, Phys. Rev. Lett. **60**, 2105 (1988).

[3]V. Kazakov, Phys. Lett. **150B**, 282 (1985); F. David, Nucl. Phys. **B257**, 45 (1985); V. Kazakov, I. Kostov, and A. Migdal, Phys. Lett. **157B**, 295 (1985).

[4]F. David, Phys. Lett. **159B**, 303 (1985); D. Boulatov, V. Kazakov, I. Kostov, and A. Migdal, Nucl. Phys. **B275 [FS17]**, 543 (1986); A. Billoire and F. David, Phys. Lett. **168B**, 279 (1986); J. Jurkievic, A. Krzywicki, and B. Peterson, Phys. Lett. **168B**, 273 (1986); J. Ambjørn, B. Durhuus, J. Fröhlich, and P. Orland, Nucl. Phys. **B270 [FS16]**, 457 (1986); D. Boulatov and V. Kazakov, Phys. Lett. B **184**, 247 (1987); V. Kazakov, Nucl. Phys. (Proc. Suppl.) **B4**, 93 (1988).

[5]V. Kazakov and A. Migdal, Nucl. Phys. **B311**, 171 (1989).

[6]I. Kostov and M. Mehta, Phys. Lett. **189B**, 118 (1987).

[7]V. Kazakov, Niels Bohr Institute Report No. NBI-HE-89-25, 1989 (to be published).

[8]M. Douglas and S. Shenker, Rutgers University report, 1989 (to be published).

[9]I. Gelfand and L. Dikii, Usp. Mat. Nauk **30**, 5 (1975).

[10]D. J. Gross and A. A. Migdal (to be published).

[11]F. Dyson, J. Math. Phys. **3**, 157 (1962).

[12]D. Bessis, Commun. Math. Phys. **69**, 147 (1979); C. Itzykson and J. B. Zuber, J. Math. Phys. **21**, 411 (1980).