Nearly Incompressible Hydrodynamics and Heat Conduction

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By means of an asymptotic analysis, two distinct approaches to incompressibility are found for a low-Mach-number ideal fluid, distinguished according to the relative magnitudes of temperature, density, and pressure fluctuations. For heat-conduction-dominated fluids, temperature and density fluctuations are predicted to be anticorrelated, and the classical passive scalar equation for temperature is recovered, whereas a generalized "pseudosound" relationship for the fluctuations is found for heat-conduction-modified fluids, together with a modified thermal equation. The full set of nearly incompressible dynamical equations is described.

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The theory of fluid[1,2] and magnetofluid[3,4] turbulence remains an active area of fundamental research[5] and provides a mathematical basis for many physical applications of current interest. Of the better developed turbulence theoretic formulations, most are restricted to the case of incompressible fluids, with a constant density and a solenoidal flow velocity. However, observational evidence in laboratory systems[6] and in astrophysical[7] and space[8,9] plasmas points to the potential importance of certain compressibility effects, especially density fluctuations, even when incompressibility appears to be an otherwise good approximation.[10,11] Accordingly, there has been considerable interest in exploring the relationship between density fluctuations and incompressible turbulence[6,7,10–12] recently, as well as the more general relationship between compressible and incompressible fluid models.[13,14] We address further these crucial issues in this Letter, including for the first time, as far as we are aware, consideration of the full ideal-gas equation of state and the effects of heat conduction. Our conclusions clarify previous results based on more restrictive assumptions,[10,12,14] extend the applicability of descriptions based on incompressible turbulence, and suggest novel experimentally observable features of certain flows having nearly constant density.

In the following, we demonstrate that the equation of heat transfer typically used in studies of incompressible turbulence[1] should be interpreted correctly as an equation of nearly incompressible hydrodynamics and not of incompressible hydrodynamics. This result is a direct consequence of applying perturbative techniques developed recently by Zank and Matthaeus[15] to the special case of thermally conducting hydrodynamics. Such perturbative techniques were developed to clarify the relationship between low-Mach-number compressible and incompressible fluids in which the limiting solution of the compressible equations satisfies a completely different nonlinear partial differential equation (PDE) as the Mach number tends to zero.[16] In summary, our approach is to derive a modified system of fluid equations which retain the effects of compressibility weakly (such as density fluctuations) yet contain the incompressible-fluid solutions as the leading-order, low-Mach-number solutions. Such an approach was initiated, for ideal polytropic compressible flows, by Klainerman and Majda,[13,17,18] who postulated a set of modified hydrodynamic equations and proved rigorously that their equations converge to the incompressible hydrodynamic equations with decreasing Mach number. For obvious reasons, we call such modified equations "nearly incompressible."[14]

To illustrate the power and generality of our approach, we investigate in detail the hydrodynamic equations with heat conduction and show that there exist two distinct sets of nearly incompressible equations, both having the incompressible hydrodynamic equations as their limiting case. The most remarkable aspect of our analysis is that the well-known equation of heat transfer for an incompressible fluid[16] arises quite naturally as an equation of nearly incompressible hydrodynamics for situations in which thermal processes dominate the fluid dynamics. Besides clarifying the standard rather unsatisfactory derivation of the "passive scalar" equation,[19] our approach also illustrates that the incompressible heat-transfer equation[1,16] is valid only when temperature and density fluctuations are significantly more dominant than the pressure fluctuations. In this case, use of the other nearly incompressible equations reveals that the density and temperature fluctuations $\delta\rho,\delta T$ are *anticorrelated,* in the sense that $\delta\rho \propto -\delta T$—a result of considerable value, both experimentally and theoretically, for heat-conduction-dominated turbulence. Alternatively, for the case when no one of the pressure, density, or temperature fluctuations dominates the others, we derive a different thermal-transfer equation modified by acoustic effects. The fluctuating quantities are related via $\delta T \propto \delta p - \delta\rho$ which, as we show below, reveals immediately that if the pressure and temperature fluctuations are correlated, then $\delta p = c_s^2 \delta\rho$ (c_s the sound speed). This is the basis of the pseudosound approximation used to relate density fluctuations to incompressible pressure fluctuations.[2,6,10,12,20]

By using the normalizations $\mathbf{x}/L$, u_0t/L, ρ/ρ_0, and p/p_0, with L the Reynolds-number length scale and p the pressure, the normalized continuity and momentum equations become

$$\partial_t\rho+\nabla\cdot\rho\mathbf{u}=0\,, \tag{1}$$

$$\rho\,\partial_t\mathbf{u}+\rho\mathbf{u}\cdot\nabla\mathbf{u}=-\epsilon^{-2}\nabla p\,, \tag{2}$$

where, for highly subsonic flows, $\epsilon^2=\gamma u_0^2/c_s^2=\gamma M_s^2\ll 1$, with γ the ratio of the specific heats. For the ideal gas, $p\propto\rho T$. Kreiss,[21] in studying symmetric hyperbolic PDE's with widely separated time scales, showed that if the solution is to vary on the slow time scale alone, then it is necessary that several time derivatives (and, in particular, at time $t=0$) of the solution be bounded of order 1. This procedure suppresses fast-scale variations such as acoustic waves and allows the solution and its derivatives to be estimated independently of ϵ. The limit $\epsilon\rightarrow 0$ can then be considered and asymptotic expansions derived. Evidently, for $\partial_t\mathbf{u}$ to be bounded independently of ϵ, it is necessary to choose the normalized pressure as $p=1+\epsilon^2p_1$. It can further be shown that $\partial_{tt}\mathbf{u}$ is bounded if and only if $\nabla\cdot\mathbf{u}=0$. Finally, from the energy equation, it can be shown that the density ρ must be constant. Thus, application of Kreiss's principle[21] yields the equations of incompressible hydrodynamics directly as constraints on the subsonic compressible hydrodynamic equations which eliminate all solutions which vary on fast time scales. This gives mathematical expression to the physically intuitive arguments commonly advanced to justify the validity of the incompressible hydrodynamic equations.[16] We denote by $\mathbf{u}^\infty$ and p^∞ the solutions of (1) and (2) which vary on slow time scales only, i.e., solutions of the incompressible hydrodynamic equations

$$\partial_t\mathbf{u}^\infty+\mathbf{u}^\infty\cdot\nabla\mathbf{u}^\infty=-\nabla p^\infty,\quad \nabla\cdot\mathbf{u}^\infty=0\,. \tag{3}$$

By employing appropriate thermodynamic identities, the equation of heat transfer can be expressed in two forms:

$$\rho(\partial_tT+\mathbf{u}\cdot\nabla T)-(\partial_tp+\mathbf{u}\cdot\nabla p)=\mathrm{Pr}^{-1}\nabla^2T\,, \tag{4}$$

$$\rho(\partial_tT+\mathbf{u}\cdot\nabla T)-\gamma p\nabla\cdot\mathbf{u}=\gamma\,\mathrm{Pr}^{-1}\nabla^2T\,, \tag{5}$$

where the normalization ρ_0C_pT/p_0 has been introduced, together with the Prandtl number Pr [$C_{p,V}$ the specific-heat capacity at constant pressure (volume), $\gamma=C_p/C_V$].[16] The standard reasoning employed in deriving the "passive scalar" thermal-transport equation from (4) and (5) is not particularly satisfactory or enlightening, and neither is the subsequent interpretation and justification of the physical content.[19]

Let us consider solutions which are weakly perturbed about the slow-time-scale solutions, i.e., $\mathbf{u}=\mathbf{u}^\infty+\epsilon\mathbf{u}_1$, $p=1+\epsilon^2(p^\infty+p^*)$, so that the fast-time-scale modes vary at worst only as $O(\epsilon^{-1})$. Some care should be exercised in choosing the scaling for the density and temperature fluctuations. Consideration of either the ideal-gas law or the "principle of least degeneracy"[22] reveals that either of the choices $\rho=1+\epsilon\rho_1$, $T=T_0+\epsilon T_1$ or $\rho=1+\epsilon^2\rho_1$, $T=T_0+\epsilon^2T_1$ is entirely consistent. The first choice corresponds to a fluid in which heat conduction *dominates* the dynamics, and the second to a heat-conduction-*modified* fluid. For highly subsonic flows, the convective time scale and the "sound crossing" time scale are widely separated except for very long wavelengths. Thus, to obtain a uniformly valid expansion, we introduce the multiple scales[22] $\tau=t$, $\tau'=\epsilon^{-1}t$ (slow and fast time scales) and $\eta=\mathbf{x}$, $\xi=\epsilon\mathbf{x}$ (short and long wavelength scales) and use either of the *Ansätze* above.

To illustrate the general procedure, we consider (3) and (4) for the heat-conduction-dominated case. To the lowest order in the nearly incompressible heat-conduction-dominated expansion of the compressible fluid equations, we can neglect the contributions of the acoustic modes. Thus, in the absence of p^*, we obtain, to the first three orders

$$\begin{aligned}&\frac{\partial T_1}{\partial\tau'}=0\quad[O(\epsilon^0)]\,,\\&\frac{\partial T_1}{\partial\tau}+\mathbf{u}^\infty\cdot\nabla_\eta T_1=\frac{1}{\mathrm{Pr}}\nabla_\eta^2T_1\quad[O(\epsilon)]\,,\\&\mathbf{u}^\infty\cdot\nabla_\xi T_1+\mathbf{u}_1\cdot\nabla_\eta T_1+\frac{\rho_1}{\mathrm{Pr}}\nabla_\eta^2T_1-\partial_\tau p^\infty\\&\qquad-\mathbf{u}^\infty\cdot\nabla_\eta p^\infty=2\,\mathrm{Pr}^{-1}\nabla_\eta\cdot\nabla_\xi T_1\quad[O(\epsilon^2)]\,,\end{aligned} \tag{6}$$

from which it can be seen that the temperature fluctuation T_1, like the density, is a function of the slow time scale alone. On combining Eqs. (6), neglecting higher-order expressions, and rewriting in terms of the original variables, we obtain the equation of heat transfer for *nearly incompressible hydrodynamics,*

$$\partial_tT_1+\mathbf{u}^\infty\cdot\nabla T_1=\mathrm{Pr}^{-1}\nabla^2T_1\,, \tag{7}$$

which, of course, is nothing more than the well-known passive scalar equation, although derived here in a completely self-consistent and clear fashion. [Note that even had we retained p^*, we would still obtain (7) at this order.] It should be recognized that our mathematical arguments are in complete accord with the usual intuitive arguments advanced to justify (7). However, our analysis can also be applied to the alternate form of the thermal-transfer equation (5). In an exactly analogous way, we can obtain another thermal-transfer equation valid in a nearly incompressible fluid,

$$\partial_tT_1+\mathbf{u}^\infty\cdot\nabla T_1+\gamma\nabla\cdot\mathbf{u}_1=\gamma\,\mathrm{Pr}^{-1}\nabla^2T_1\,. \tag{8}$$

Thus, for (7) and (8) to be compatible we require that the velocity fluctuations $\mathbf{u}_1$ satisfy the nonsolenoidal equation

$$\nabla\cdot\mathbf{u}_1=(\gamma\mathrm{Pr})^{-1}(\gamma-1)\nabla^2T_1\,. \tag{9}$$

The final nearly incompressible equations come from the continuity and momentum equations,

$$\partial_t \rho_1 + \mathbf{u}^\infty \cdot \nabla \rho_1 + \nabla \cdot \mathbf{u}_1 = 0, \tag{10}$$

$$\partial_t \mathbf{u}_1 + \mathbf{u}^\infty \cdot \nabla \mathbf{u}_1 + \mathbf{u}_1 \cdot \nabla \mathbf{u}^\infty - \rho_1 \nabla p^\infty = 0. \tag{11}$$

Equations (7)-(11) represent a new system of fluid equations, nearly incompressible heat-conduction-dominated hydrodynamics, and we emphasize that these equations are not merely of academic interest. They provide, first of all, a self-consistent set of equations, simpler than (1), (2), and (4), which allows us to study, both numerically and analytically, compressible effects in low-Mach-number fluids in terms of the core incompressible-fluid equations. Second, simple manipulation of (7), (9), and (10) yields at once the relation

$$\gamma \rho_1 = -(\gamma - 1) T_1, \tag{12}$$

indicating that the two fluctuations are anticorrelated. As noted above, this result contrasts strongly with the pseudosound relation.

The implications of our analysis are rather different for the case of heat-conduction-*modified* hydrodynamics since, unlike the previous case, acoustic modifications are present in the nearly incompressible equations. This is reflected in both the thermal-transport equation and the momentum equation. Indeed, the two forms of the nearly incompressible thermal-transport equation are found to be

$$\partial_t T_1 + \mathbf{u}^\infty \cdot \nabla T_1 - \mathrm{Pr}^{-1} \nabla^2 T_1 - \partial_t p^* - \mathbf{u}^\infty \cdot \nabla p^* = \partial_t p^\infty + \mathbf{u}^\infty \cdot \nabla p^\infty, \tag{13}$$

$$\partial_t T_1 + \mathbf{u}^\infty \cdot \nabla T_1 + \epsilon^{-1} \nabla \cdot \mathbf{u}_1 = \gamma \mathrm{Pr}^{-1} \nabla^2 T_1,$$

from which we obtain the compatibility condition

$$\partial_t p^* + \mathbf{u}^\infty \cdot \nabla p^* - (\gamma - 1) \mathrm{Pr}^{-1} \nabla^2 T_1 + \epsilon^{-1} \nabla \cdot \mathbf{u}_1 = -\partial_t p^\infty - \mathbf{u}^\infty \cdot \nabla p^\infty. \tag{14}$$

It can be shown that on fast time and short wavelength scales, p^* satisfies an acoustic wave equation and may therefore be identified as the acoustic contribution to the total pressure. Furthermore, (14) reveals that the incompressible-fluid fluctuations act as a source of acoustic waves.[20] One can further show that the incompressible turbulence drives low-frequency, long-wavelength modes which, had viscosity been included in (2), are damped viscously and thermally. Thus energy, besides cascading down from large eddies to small in the incompressible fluid, is also transferred from eddies to damped long-wavelength acoustic modes. The model is closed by the modified momentum and continuity equations,

$$\partial_t \mathbf{u}_1 + \mathbf{u}^\infty \cdot \nabla \mathbf{u}_1 + \mathbf{u}_1 \cdot \nabla \mathbf{u}^\infty = -\epsilon^{-1} \nabla p^*, \tag{15}$$

$$\partial_t \rho_1 + \mathbf{u}^\infty \cdot \nabla \rho_1 + \epsilon^{-1} \nabla \cdot \mathbf{u}_1 = 0. \tag{16}$$

Observe that the equations of nearly incompressible heat-conduction-modified hydrodynamics are linear about the incompressible flow, thus making them relatively tractable. It is easily seen that use of (13)-(16) yields

$$p^* + p^\infty - T_1 = -\gamma^{-1} T_1 + \rho_1. \tag{17}$$

Thus, if the acoustic and incompressible pressure fluctuations are correlated with the temperature fluctuations, then it follows that $p^* + p^\infty = c_s^2 \rho_1$ in non-normalized terms. Therefore, it is apparent that the pseudosound theory[6,10,12,14] relies on the assumption that temperature and density fluctuations are correlated, though we defer further examination of the validity of this assumption to a subsequent report.[15]

Since the nearly incompressible heat-conduction-dominated model and the pseudosound model lead to such dramatically different results with respect to the density and temperature correlations, it is clear that the choice of which model to apply to a given situation is most critical and requires detailed assumptions about which physical processes are dominant. Evidently, interpretation of solar-wind[8-10] and interstellar data[7] favors the heat-conduction-modified, pseudosound picture. It remains to be seen if other observations of low-Mach-number gases or plasmas will reveal the correlations (12) associated with the heat-conduction-dominated limit. Other questions are also raised by the present discussion. One issue is whether there are physical applications in which importance might be attached to the distinction between the purely passive scale behavior of the temperature fluctuations in the heat-conduction-dominated case and the distinct temperature behavior obtained for heat-conduction-modified flows. Finally, further consequences and properties of the two distinct sets of ideal-gas nearly incompressible dynamical equations warrant investigation, including mathematical issues such as rigorous convergence to incompressibility,[13,17,18] and further study of the dynamical properties of nearly incompressible turbulence.

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