Cosmic-String Evolution: A Numerical Simulation

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A system of intercommuting cosmic strings is numerically evolved in a homogeneous radiation-dominated expanding universe. Further evidence for the existence of a "scaling" solution is presented, and results for the long-string energy density are given. Long strings exhibit significant small-scale structure—kinks and short-wavelength propagating modes. Net loop production is observed to be strongly peaked at scales considerably smaller than the horizon, and reasons for this severe fragmentation are briefly discussed. The new numerical techniques employed are also described.

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It has been suggested that vortex strings might appear at a very early phase transition in the Universe.[1] The evolution of such cosmic strings is of considerable astrophysical interest,[2-4] and their existence could rule out or constrain certain particle-physics models. In the usual picture, cosmic strings intercommute[5] to form closed loops, which then lose energy through gravitational radiation.[3] For the model to succeed, this energy loss must prevent strings from dominating the Universe. Analytic[6-7] and numerical[8-10] work by previous groups has shown that the energy density remaining in the string network approaches a small constant fraction of the total energy density of the Universe. However, there is disagreement over the value of this energy density, and on the mechanisms by which a "scaling" solution is approached. This paper presents results of an independent numerical simulation, and resolves some of the differences between the simulations previously carried out by Albrecht and Turok[8,10] and by Bennett and Bouchet[9,11] (AT and BB, respectively).

In a conformally flat space with metric $ds^2=a^2(\tau)\times(-d\tau^2+d\mathbf{x}^2)$, a cosmic string is described by a function $\mathbf{x}(\tau,\sigma)$, which in the gauge $\dot{\mathbf{x}}\cdot\mathbf{x}'=0$ obeys the equation of motion[12]

$$\ddot{\mathbf{x}}+2\frac{\dot{a}}{a}\dot{\mathbf{x}}(1-\dot{\mathbf{x}}^2)=\frac{1}{\epsilon}\left(\frac{\mathbf{x}'}{\epsilon}\right)', \tag{1}$$

where the primes denote $d/d\sigma$. The energy of the string is given by $\mu a(\tau)\int\epsilon\,d\sigma$, where $\epsilon^2=\mathbf{x}'^2/(1-\dot{\mathbf{x}}^2)$, and μ is the mass per unit length of the string. Equation (1) implies that the linear energy density evolves in time as $\dot{\epsilon}/\epsilon=-2\dot{\mathbf{x}}^2\dot{a}/a$.

The numerical problems arising from this equation of motion have been delineated by BB. Beyond the nonlinearity of Eq. (1), these are due to the velocity and string direction discontinuities (or "kinks") which are introduced when two strings intercommute. These kinks build up over time (every time strings intercommute, four kinks are created) so the numerical scheme must evolve them accurately.[9] Unless the scheme is specially tailored to handle such contact discontinuities, it is well known that over a period of N time steps the width of the transition typically spreads as $N^{1/(R+1)}$, where R is the order of accuracy of the numerical scheme.[13] AT have approached this problem with numerical viscosity which spreads out the kinks, and in their original work BB used preprocessing (initial smearing).

Our evolution algorithm differs from these methods. It uses a technique drawn from the literature on hyperbolic conservation methods, which is designed for solving problems involving shocks.[13] Defining coordinates $\boldsymbol{\alpha}=\mathbf{x}'-\epsilon\dot{\mathbf{x}}$ and $\boldsymbol{\beta}=\mathbf{x}'+\epsilon\dot{\mathbf{x}}$, the equations may be rewritten in first-order form as

$$\dot{\boldsymbol{\alpha}}=-\left(\frac{\boldsymbol{\alpha}}{\epsilon}\right)'-\frac{\dot{a}}{a}(\boldsymbol{\beta}-\boldsymbol{\alpha}), \tag{2}$$

$$\dot{\boldsymbol{\beta}}=\left(\frac{\boldsymbol{\beta}}{\epsilon}\right)'-\frac{\dot{a}}{a}(\boldsymbol{\alpha}-\boldsymbol{\beta}), \tag{3}$$

$$\dot{\epsilon}=-\frac{\dot{a}}{a}\epsilon^{-1}(\epsilon^2-\boldsymbol{\alpha}\cdot\boldsymbol{\beta}), \tag{4}$$

$$\dot{\mathbf{x}}=\frac{1}{2\epsilon}(\boldsymbol{\beta}-\boldsymbol{\alpha}). \tag{5}$$

Given the relatively slow expansion rate, the right- and left-moving modes $\boldsymbol{\alpha},\boldsymbol{\beta}$ propagate in a variable energy background with velocities ϵ^{-1} and $-\epsilon^{-1}$, respectively. The key point is that in flat space ($\dot{\epsilon}=0$) the system (2)–(5) is in *conservation form*.

After extensive trials, we adopted a high-resolution total-variation-nonincreasing (TVNI) algorithm.[13] The scheme uses a five-point wide stencil to evaluate spatial derivatives, and can be proven to be second-order accurate. The algorithm will be described in greater detail elsewhere; it is related to artificial compression methods which use analytic estimates of the smearing rate of

discontinuities to apply appropriate nonlinear corrections, thus avoiding the use of viscosity. The string position is updated from Eq. (5) with an interleaved two-step Runge-Kutta algorithm.

Numerical experiments show that kinks spread out to between three and five string points during the first few time steps and then remain at this width for hundreds of loop oscillations. The periodicity of small loops is preserved to high accuracy. Evolution and energy conservation are satisfactory for loops having more than sixteen points, and excellent above twenty points (throughout the simulations we adopt a lower cutoff N_s of at least sixteen points). Simple analytical solutions in a radiation-dominated expanding universe are also accurately reproduced.

The second computational difficulty is that of locating string segments which cross. We implement the optimal solution,[14] which is to break up the space (a three torus) into $\mathcal{N}^3$ boxes where $\mathcal{N}=(\text{box size})/|\Delta\mathbf{x}|$. A typical value of $\mathcal{N}$ is of the order 10^3. Since $\mathcal{N}^3$ is then very large, it is not practical to have an array of 10^9 pointers to linked lists of points occupying the cell in question. However, since most cells are empty, the array only contains entries for occupied cells. Thus crossing detection proceeds in two stages. An ordered list of all occupied cells is constructed, with pointers to a linked list of the points in those cells. One then steps through each of the occupied cells, looking for crossings between its segments and those in neighboring cells. This is done using the method described by BB, where one looks for a change in the sign of the tetrahedron formed by two segments. The intercommutation is not allowed if it would form a loop of less than N_s string points.

If this condition is satisfied, the affected points are time stepped backwards to the predicted crossing time. Spline interpolants (functions of σ) with six points taken from each string segment are used to locate the crossing points. The strings are then locally reparametrized by shifting points and rescaling ϵ along the interpolant, to ensure that the crossing takes place halfway between the new string points. The "previous" and "next" pointers of the two affected segments are then interchanged, and the time-step routine is used to advance the points forward in time to their original values. This procedure is better than just exchanging pointers, and is also better than linearly redistributing energy among the affected points, because the jumps in energy between points are reduced by spreading them more widely.

In principle, the overall searching-crossing algorithm has a computation time given by $ap+bp\ln p$, but in practice, for $p\approx 10^6$, the linear term dominates. The efficiency of this algorithm can be appreciated from the fact that it uses far less computation time than the time-stepping routine.

The initial conditions are a random walk of correlation length ξ_0, generated by the method of Vachaspati and Vilenkin[15] with three slight modifications. First, the corners that appear in the random walk are rounded. The resulting network is composed of either straight or curved segments, with N_ξ points per straight segment and $[(\pi/4)N_\xi+\frac{1}{2}]$ points per round corner segment (where [ ] is the integer part). The second modification is that nonzero initial velocities are assigned to the points that make up the string. Because the average velocity squared of string loops in flat space is one-half, this is more realistic. The initial velocity has a constant magnitude $1/\sqrt{2}$ at each point on the string, lies in a plane orthogonal to the direction of the string, and rotates by a random angle chosen between $-\pi$ and π as one moves along the string between successive points on the ξ_0 grid, as shown in Fig. 1. The third modification is that the strings are assigned a random orientation, so that the total winding numbers are not necessarily zero.

Throughout the runs the following quantities are monitored to check the accuracy of the program: (i) Energy conservation for both left and right movers (e.g., $\sum_k |\,|\boldsymbol{\alpha}_k|/\epsilon_k\,|$). During the course of a simulation this remains above 85%, after up to four expansion times. Most of this energy loss can be directly accounted for by the initial four-point smoothing of kinks after intercommuting. The perpendicularity gauge constraint is also satisfactorily preserved. These constraints are much more tightly conserved for the long strings ($\geq 95\%$), the properties of long strings being the primary concern of this paper. (ii) Position consistency between $x'_k=\frac{1}{2}(\beta_k+\alpha_k)$ and the spatial derivative obtained from the positions $(x_{k+1}-x_{k-1})/2\Delta\sigma$ also remains within about 3% throughout. This is an important confirmation of the accuracy of our position evolution, which is not dependent on a reconstruction using the x'.[10,11] (iii) Loop mismatch $\oint x'\,d\sigma$ was also checked and always remained small (less than 1%). The fact that these quantities remain small over thousands of time steps is good reason to have confidence in the numerical methods employed.

To discuss the results of our radiation-era simulations, time and length are measured in units of the initial horizon length H_0 at the start of the simulation. Physical time is the proper time elapsed since the initial singularity, $t=\int a(\tau)d\tau$, and the horizon length is $H(t)=a(\tau)\int d\tau=2t$. A long (short) string is defined as one whose total energy is greater (less) than $\mu\pi t$. The simulations begin at time t_0, where $t_0/H_0=1/2$. The scale factor is $a(t)=(2t/H_0)^{1/2}$, and the initial-string density is determined by the ratio $\alpha=H_0/\xi_0$ of the horizon length to the correlation length of the initial-string network. In principle, the length of the runs is restricted by the number of initial correlation-length cells because the horizon becomes larger than the size of the numerical box after sufficient expansion.

For initial-string velocities equal to 0 and $1/\sqrt{2}$, the simulations approach similar long-string densities at late times, although the final density of the initial static net-

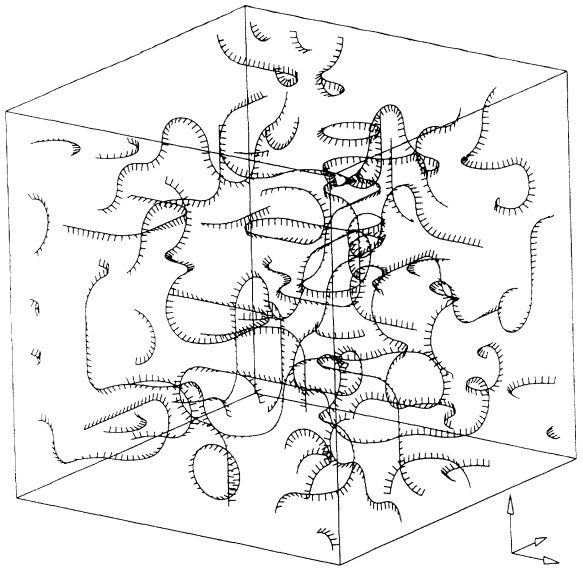

FIG. 1. Vachaspati-Vilenkin initial conditions in a small box split into 5^3 correlation cells. These are supplemented by a random velocity winding along the string, which is indicated by the small lines normal to the string direction.

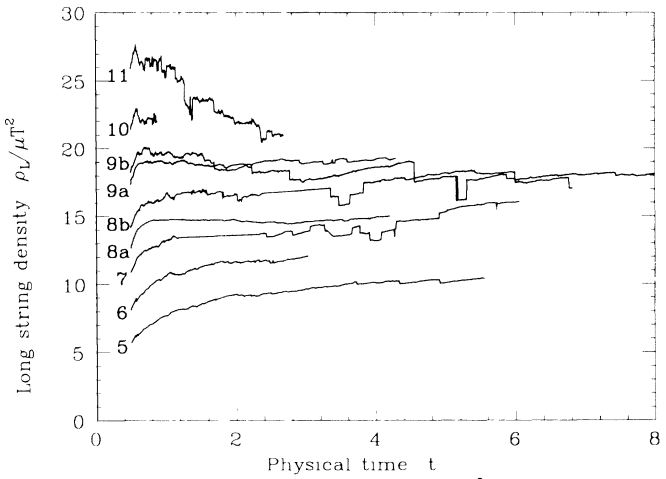

FIG. 2. The long-string energy density $\rho_L t^2/\mu$. The runs are labeled by α; see Table I for the parameters of these runs.

work tends to be about slightly higher ($\leq 5\%$). The initially static network takes longer to relax toward scaling, partly because it takes time for the static strings to accelerate up to randomized relativistic speeds. For this reason, an initial velocity of $1/\sqrt{2}$ is used throughout. For comparison with previous work, note that the additional kinetic energy makes the energy density higher for the same initial ratio H_0/ξ_0, for example, $(H_0/\xi_0)_{BB} \rightarrow 2^{1/3}(H_0/\xi_0)$ to obtain the same initial energy density.

The parameter that measures the size of the cutoff is $\lambda = N_s/N_\xi$. The effect of decreasing λ was investigated by increasing N_ξ for $N_s = 16$. In agreement with BB, we found that the effect on the final long-string energy density diminished as λ was reduced towards 0.5. We thus used $\lambda < 0.6$ for our long runs.

Figure 2 and Table I present the results of several short and long runs. The short runs were used to bracket the scaling solution. In these runs, the initial-string density α was varied for a fixed set of initial conditions. One can see that the long-string energy density is attracted to a constant value of $\rho t^2/\mu$ in the range from 10 to 22, for widely differing values of α.

The parameters of the long runs are given in Table I. During these runs, the Universe expands in spatial volume by a factor of over 4^3. The final-string density approaches $\rho t^2/\mu = 16 \pm 4$ at late times, though systematic effects related to the small-loop cutoff could reduce it slightly further. This is in good agreement with the work of BB, who quote[9] a value of 20 ± 10 (and more recently[11] 13.0 ± 2.5), and in apparent disagreement with AT, who find[10] a value of 50 ± 25. The sudden rises (drops) in the density are associated with string loops which suddenly move out of (into) the horizon or, more precisely, our definition of the long-string cutoff. They can be adequately averaged out by eye. The energy density in small loops, as expected, scales approximately like that of matter. In reality, these loops would gradually lose their energy by gravitational radiation, over a time scale much longer than that of the present simulations.

Finally, we comment on some qualitative features of the evolved string network as shown in Fig. 3. A box of

TABLE I. The parameters for five of the longest runs shown in Fig. 2. The initial configuration has (cells)^3 correlation cells. The ratio of scale factors gives the expansion of the Universe. The CPU time is on a VAX-8550 computer, except for the low-resolution run 8*a* which gives an approximate Cray-2 time.

Run	α	Cells	$\frac{a(t_f)}{a(t_0)}$	Time steps	N_ξ	N_s	Points	CPU time
7	7	22	3.5	1540	28	16	218628	230
8*a*	8	50	2.9	650	14	8	630000	24
8*b*	8	22	3.7	2001	28	16	218628	320
9*a*	9	35	3.0	1300	28	16	859602	200
9*b*	9	22	4.2	3100	30	18	237858	350

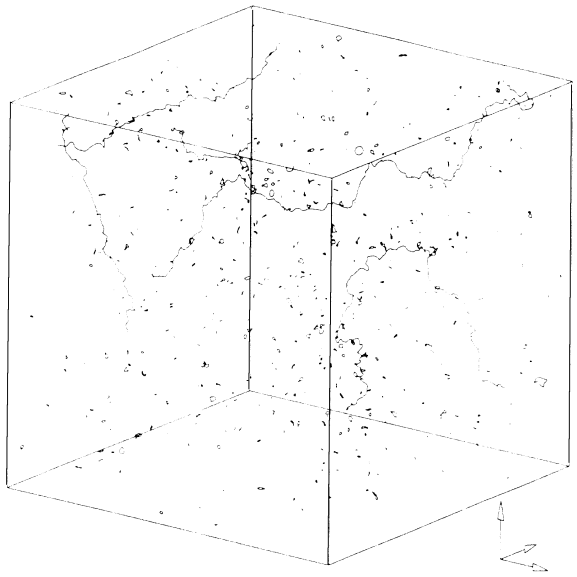

FIG. 3. The evolved cosmic-string system, showing a cube with side length a fraction 0.28 of the horizon.

side length one-quarter of the horizon is illustrated after evolution for over four expansion times. The preponderance of small-scale structure—small loops, kinks, and short-wavelength propagating modes—is immediately apparent, despite the much larger overall long-string correlation length. Loops are principally created on scales much smaller than the horizon; at sizes determined by the resolution of the present simulations. Since the minimum loop size is a fixed physical scale, by the end of the longest simulations energy production is strongly peaked around loops of radius $\leq 10^{-2}H$. Unlike the long strings, small-loop production is not yet observed to be scaling. We are currently investigating reasons for this severe fragmentation,[16] but preliminary results indicate that low coherent velocities $\langle v \rangle \sim 0.15$ on the network correlation length are an important factor. Such low velocities imply the collapse of large loops and curved regions of string, the relative enhancement of substructure, and an accompanying increase in the likelihood of self-intersections. Video animations of the evolving string system are especially revealing; reconnections between uncorrelated long strings result in significant small-loop production by this process. The importance of "kinkiness" and small loops has been emphasized by BB, but it seems Fig. 3 exhibits even more small-scale structure than their results[9] (possibly because of the better preservation of kinks by our numerical scheme).

Prior to this work, two possible explanations had been suggested for the disparate findings of BB and AT. One was that the implicit numerical viscosity used by AT damped out the kinks on the long strings, thus suppressing the production rate for very small loops. The other possibility was that the evolution algorithm used by BB was numerically unstable and spontaneously generated kinks, resulting in an over production of small loops. Our simulations tend to favor the former explanation.

Beyond these characteristics of cosmic-string evolution, some important quantitative questions remain unanswered, particularly with regard to the observational consequences deriving from small-loop production. The efficient and accurate numerical code we have developed can easily be pushed to higher-resolution and larger simulations, and these issues are being pursued vigorously.[16]

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