Phase Slip and Turbulence in Superfluid ^{4}He: A Vortex Mill That Works

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It is shown that superfluid turbulence arises as a convective, not an absolute, instability. A continuous source of macroscopic quantized vortices is therefore required, not only to initiate the turbulence, but also to keep the vortex tangle alive. It is demonstrated that a streamwise-pinned, remanent vortex at the channel inlet can act as the fluid-dynamical analog of a phase-slip center, injecting vortex filaments into the flow at a steady rate. Several such vortex mills in tandem are sufficient to initiate and sustain the turbulent state.

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Superfluid ^{4}He moves without friction at low velocities. Upon exceeding some critical velocity, however, it enters a dissipative, microscopically turbulent state consisting of a self-sustaining tangle of quantized vortex lines.[1,2] How this dynamical state is initiated and sustained is not well understood, and forms the topic of this paper. We first discuss why a continuous vortex source is required, and then propose a possible model for one.

The physics governing individual quantized vortices appears to be well characterized by the statement that they behave like vortex filaments in an ideal fluid, while also experiencing a frictional force due to the motion of elementary excitations past the vortex. A good description of the motion of an individual vortex $\mathbf{s}(\xi,t)$ is then provided by the localized induction approximation:

$$\dot{\mathbf{s}}=\mathbf{v}_s+\beta\mathbf{s}'\times\mathbf{s}''+\alpha\mathbf{s}'\times(\mathbf{v}_n-\mathbf{v}_s-\beta\mathbf{s}'\times\mathbf{s}''), \tag{1}$$

where $\mathbf{v}_n$ and $\mathbf{v}_s$ are the applied macroscopic normal and superfluid velocities, α is a temperature-dependent friction constant, and $\beta=(\kappa/4\pi)\ln(cR/a_0)$, with κ the quantum of circulation, R the characteristic radius of curvature of the filament, c a constant of order 1, and a_0 the core radius of the vortex. The primes denote differentiation with respect to the arc length. In order to incorporate the most important effect of the interactions between the vortices, it is necessary to add the condition that lines which cross will undergo a reconnection.[3] Because vortex loops can now be pinched off from vortices already present, this makes possible the multiplication of vortex lines and thus the development of the vortex tangle.

The reconnecting-vortex model by itself does not explain the initiation of superfluid turbulence, since it has no provision for creating vortex singularities from scratch. To meet this difficulty, it has often been suggested[4-6] that superfluid ^{4}He always contains at least some metastably pinned, remanent vortices which serve as initiators of the vortex population explosion. This idea is also supported by a limited amount of experimental evidence.[7-9] The speculation then has been that at some critical velocity such vortices pinned across the channel will turn into *vortex mills,* i.e., sources fixed in space which spool a continual stream of quantized vortices out into the fluid. In reality, however, such pinned vortices will simply hop off their pinning sites at a rather low depinning velocity, and will be washed out of the channel.[10] Thus no working vortex mill has ever been devised, and the relevance of this idea to the initiation of turbulence has remained uncertain.

While the above difficulty is well known, it can be shown that the problem is much more general. Consider turbulence driven by pure superflow, the situation in which experimentally it is the easiest to produce turbulence in a channel. In previous work,[3] the reconnecting-vortex model has been implemented in numerical calculations where infinitely long channel behavior is simulated by applying reentrant boundary conditions to some section of the channel, vortices which leave the downstream face reentering the computational volume on the upstream face. Not only is a self-sustaining vortex tangle obtained with properties that are in excellent agreement with experiment, a critical velocity $v_c(\infty)$ below which the tangle ceases to be self-sustaining is also found.[11] As is obvious from the first term on the right-hand side of Eq. (1), however, the vortex tangle experiences an overall drift velocity v_s. In a real channel, therefore, the question arises whether the vortex tangle can sustain itself against this tendency to flush the vortices bodily downstream. To study this problem we have computed the development of finite plugs of turbulence, at values of v_s for which the reentrant (infinite-channel) calculations give self-sustaining turbulence. The result is always as illustrated in Fig. 1: the turbulent region indeed grows, the plug getting wider as it propagates, but this process is much slower than the rate at which the tangle is washed downstream. At least within the context of the reconnecting-vortex model, therefore, the internal dynamics of the tangle is such that any vortex tangle of finite extent will be flushed away.

In the language of fluid mechanics, the aforegoing amounts to the finding that, beyond $v_c(\infty)$, the superfluid is *convectively* unstable against the growth of a vor-

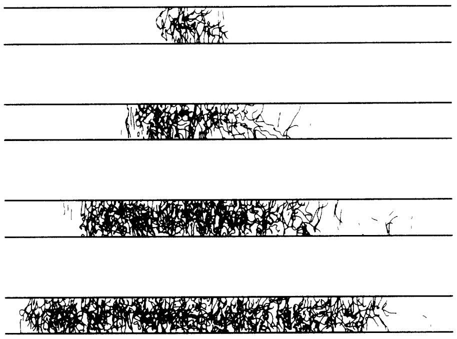

FIG. 1. Evolution of a turbulent plug at a velocity above $v_c(\infty)$. The calculations were carried out in a smooth, circular channel using $\alpha=0.1$, and $v_s=55\beta/D$, D being the diameter of the channel. The figure tracks on the center of the plug, which has moved (from top to bottom) a distance 0, $40D$, $80D$, and $120D$ down the channel.

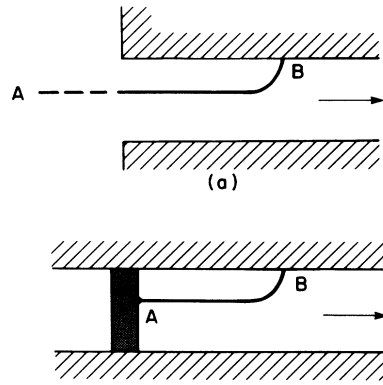

FIG. 2. New remanent-vortex geometries arising from the consideration of end effects.

tex tangle, but that it is never *absolutely* unstable. To maintain the superfluid turbulent state therefore requires the continuous injection of a sufficiency of dynamically active quantized vortices at the channel inlet to allow the fully tubulent state to develop downstream. *Therefore, a steadily functioning vortex mill is required not just for the initiation, but also for the continued existence of superfluid turbulence.*

We have found that a working vortex-mill model suggests itself naturally once one begins to consider the inlet region of the flow channel. Consider remanent vortices which have become dynamically active as shown in Fig. 2. End B feels the full tangential field v_s in the channel and is (in the illustration) being flushed downstream. End A, on the other hand, feels a negligible tangential velocity, either by reason of its remoteness [Fig. 2(a)] or because of the way it is pinned [Fig. 2(b)]. In contrast to a vortex wholly internal to the channel, such a vortex will remain pinned at A, the vortex will not be flushed away, and, since end B is being washed downstream, vortex line is continually being spooled into the channel.

The behavior of such a streamwise-pinned vortex can be explored by implementing Eq. (1) numerically. For simplicity, we limit ourselves to the case of Fig. 2(b), fixing end A somewhere on the cross section of the channel and allowing B to move freely along the boundary. After some transient motions, the vortex takes on the form of a growing helix with a slowly varying wave vector $\mathbf{k}$ [Fig. 3(a)]. The sense of the helix is retrograde with respect to the circulation of the vortex, and its motion is as though it were rotating as a rigid object in the prograde direction, with a fixed angular velocity ω. In reality, a given point on the vortex advances in the flow direction with a velocity ω/k, while at the same time it moves slowly outward. This behavior is modified when the vortex nears the wall of the channel. What happens there is complicated and depends to some extent on the location of A, on the geometry of the channel, and on the degree of surface roughness. Generically, however, it is clear from Fig. 3(b) that a succession of vortex loops is injected into the flow as the end of the spiral vortex repeatedly breaks off by reconnecting to the surface. These loops are of the correct size and orientation to be amplified by the driving field, and will subsequently grow across the channel. Thus the spiral helix acts as the fluid-dynamical analog of a phase-slip center, injecting a

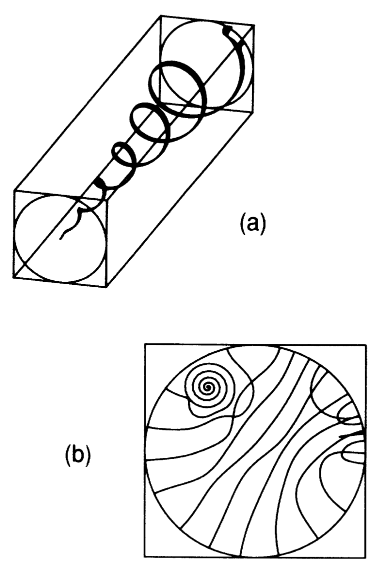

FIG. 3. (a) Steady-state spiral-helix configuration of a streamwise vortex filament pinned at the center ($\alpha=0.1$, $v_s=20\beta/D$). The figure actually consists of eight closely spaced sequential configurations. (b) End-on view of a phase-slip center in action ($\alpha=0.1$, $v_s=55\beta/D$). The vortex is pinned near the upper left. The outwardly growing spiral periodically reconnects to the boundary and releases a new line segment, which then moves to the lower right. A vortex-loop reflection (Ref. 12) occurs as the segment approaches the right end of the channel.

continuous stream of vortex segments into the flow. For smooth walls and simple channel geometries, the release rate is closely periodic.

The initial portion of the spiral helix is unaffected by what happens downstream, leading one to conclude that it represents an intrinsic instability of the streamwise-pinned vortex,[13] the growth of which is eventually cut off by the channel walls. It is an interesting problem to consider how the elegant limit-cycle behavior (illustrated in Fig. 3) arises within the context of Eq. (1), and in particular how the initial wave vector $\mathbf{k}_0$, the frequency ω, and the rate at which the helix grows with downstream distance are determined. The differential equation describing the vortex configuration is

$$\frac{\partial \mathbf{s}}{\partial t}=\dot{\mathbf{s}}-\frac{\partial \mathbf{s}}{\partial \xi}\dot{\xi}, \tag{2}$$

where $\dot{\mathbf{s}}$, given by Eq. (1), is the velocity of a particular point on the line, and $\dot{\xi}$ is the rate at which the parameter ξ specifying a particular point on the line changes because of the vortex motion. Even though the resulting set of equations for the three components of $\mathbf{s}$ is highly nonlinear, the time dependence separates out if a uniformly rotating solution of the form $y=s_2+is_3=g(s_1)\times\exp(i\omega t)$ is substituted. Here, s_1 is the downstream distance. One is left with a pair of very complicated, coupled, nonlinear ordinary differential equations for the real and imaginary parts of $g(s_1)$. Some insight can be gained by going to the small-amplitude limit, where Eq. (2) takes a linearized form leading to $g(s_1)=A\times\exp[-i(ks_1+\phi_0)]$ and the condition

$$\omega=v_s k-\beta k^2-i\alpha v_s k\,. \tag{3}$$

If one assumes that near $s_1=0$ the system picks the mode with the maximum growth rate, restricted by the condition that the group velocity of the mode still be in the downstream direction, one obtains $k_0=v_s/2\beta$ and $\omega=v_s^2/4\beta$. Although the argument leading to this prediction is by no means conclusive, the steady-state helices obtained in the simulations actually agree very closely with this prediction. The spatial growth is observed to depend only on the friction constant α and to be initially linear with downstream distance. These, and other properties of the spiral helix, remain to be investigated.

A single vortex is not sufficient to initiate and sustain superfluid turbulence. As shown in Fig. 4(a), the vortex segments are too sparse to interact: they will move along as individuals and eventually annihilate on the channel walls. Three streamwise-pinned vortices, however, are already enough, injecting vortex-line segments at a sufficient rate to engender a self-sustaining vortex tangle reaching infinitely far downstream. Although the calculations are limited to finite-channel lengths, there is no doubt that the situation shown in Fig. 4(d) represents such a case. Any section of this particular tangle, if followed downstream as in Fig. 1, is self-sustaining and reaches limiting properties which are essentially the same as those obtained in the reentrant, infinite-channel calculations reported previously. The critical velocity at which the entire channel suddenly becomes filled with turbulence is difficult to calculate accurately, again because it is computationally expensive to look very far down the channel. Referring to Figs. 4(b)-4(d), however, it is obvious that when $v_s=40\beta/D$, the initial disturbance provided by the vortex mill dies out, whereas at $v_s=55\beta/D$ it is self-sustaining down the channel. Figure 4(c) may be interpreted as a marginal case which can only be resolved by increasing the size of the computation. We find further that the strength of the vortex mill sustaining the tangle (i.e., the number of configuration of the streamwise vortices involved) has no effect on the critical velocity or on the properties of the turbulence far downstream from the source.

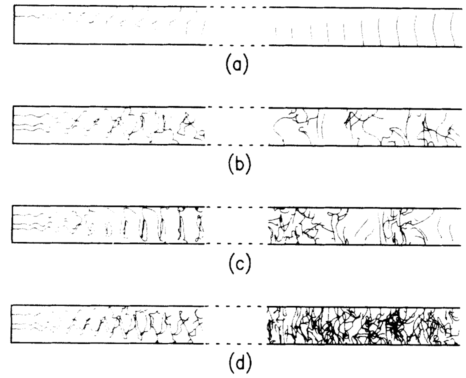

FIG. 4. Snapshots of vortex configurations in a smooth, circular channel, showing the inlet behavior over a distance of 0 to $5D$, and the behavior farther downstream in the interval $25D$ to $30D$. The conditions are (a) one vortex, $\alpha=0.1$, $v_s=55\beta/D$; (b) three vortices, $\alpha=0.1$, $v_s=40\beta/D$; (c) three vortices, $\alpha=0.1$, $v_s=47.5\beta/D$; (d) three vortices, $\alpha=0.1$, $v_s=55\beta/D$.

The demonstration that streamwise-pinned vortices washing into the channel undergo an instability which allows them to act as vortex mills, sustaining superfluid turbulence above the critical velocity $v_c(\infty)$, offers the first substantiation of the vortex-mill hypothesis advanced long ago by Feynman and others. From a more modern perspective, we have shown by explicit calculation that the reconnecting-vortex model combined with the notion of remanent vorticity is capable of accounting for the initiation and continued existence of superfluid turbulence. Aside from resolving this important conceptual issue, the model for the initiation of superfluid turbulence presented here suggests intriguing, although still speculative, interpretations of various well-known onset phenomena. For example, the common observation that it is possible to go far above $v_c(\infty)$, and then initate the turbulent state by tapping on the experiment, can plausibly be ascribed to the difficulty of breaking loose the remanent vortices required to operate the mill. Upon decreasing the driving velocity once the system is turbulent and the vortex mill activated, the system is expected to stay turbulent until $v_c(\infty)$ is reached, as is indeed observed. The vortex mill can in fact stay active down to velocities at which the downstream end repins. Because these velocities are generally much smaller than $v_c(\infty)$, the possibility exists of precursor effects, involving dissipation at a much lower level than that characteristic of well-developed turbulence. This may be relevant to the interpretation of the TI-TII transition,[2] where a region of relatively low activity is observed to precede the onset of fully developed turbulence. Finally, the finding that, above the depinning velocity, a single quantized vortex can act as a relaxation oscillator generating very regular phase-slip events provides new possibilities for the interpretation of recent experiments[14] on the flow of superfluid ^{4}He through microscopic orifices, in which just such a periodic phase-slip behavior has been observed.

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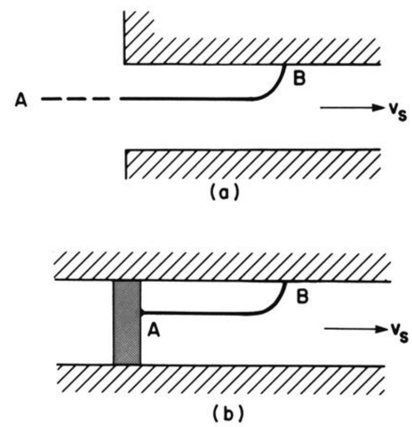

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