PHYSICAL REVIEW

LETTERS

VOLUME 63 | 28 AUGUST 1989 | NUMBER 9

Routes to Chaotic Scattering

S. Bleher,[(1),(a)] E. Ott,[(1,2)] and C. Grebogi[(1)]

[(1)]*Laboratory for Plasma Research, University of Maryland, College Park, Maryland 20742*

[(2)]*Departments of Electrical Engineering and of Physics, University of Maryland, College Park, Maryland 20742*

(Received 30 May 1989)

The onset of chaotic behavior in a class of classical scattering problems is shown to occur in two possible ways. One is abrupt and is related to a change in the topology of the energy surface. The other arises as a result of a complex sequence of saddle-node and period-doubling bifurcations. The abrupt bifurcation represents a new generic route to chaos and yields a characteristic scaling of the fractal dimension associated with the scattering function as $[\ln(E_c-E)^{-1}]^{-1}$, for particle energies E near the critical value E_c at which the scattering becomes chaotic.

PACS numbers: 05.45.+b, 03.20.+i, 03.65.Nk

We consider a classical dynamics scattering problem for a potential $V(\mathbf{r})\geq 0$ with $\lim_{|\mathbf{r}|\to\infty}V(\mathbf{r})=0$. It has recently been found that the outgoing trajectories (i.e., the trajectories after scattering) can have very complicated behavior as a function of incoming trajectories and that this results from the presence of chaotic dynamics on a fractal set in the phase space.[1] Examples where this chaotic scattering phenomenon is important include chemical reactions,[2] fluid dynamics,[3] the three-body gravitational problem,[4] and others. In addition, there are striking consequences of chaos in a classical scattering problem for the corresponding quantum problem.[5]

As an example, consider the two-dimensional potential $V(\mathbf{r})=x^2y^2\exp[-(x^2+y^2)]$. (This potential has square symmetry with four hills of equal heights $E_m=e^{-2}$ located at $x=\pm 1$ and $y=\pm 1$.) Figure 1 shows the asymptotic (for large $|\mathbf{r}|$) scattering angle ϕ ($\cos\phi\equiv\mathbf{x}_0\cdot\mathbf{p}/|\mathbf{p}|$, where $\mathbf{p}$ is the momentum) versus impact parameter b for particles incident in the direction parallel to the x axis. In Fig. 1(a) we see that the scattering function is a smoothly varying curve when the energy $E=1.626E_m$. For $E=0.26E_m$, however, the scattering is chaotic, and we see that the scattering function behaves wildly in certain regions [Fig. 1(b)] and that this wild behavior apparently persists on arbitrarily small scale [see the blowups in Figs. 1(c) and 1(d)]. In fact,[6] for $E/E_m=0.260$, the scattering function is singular on a fractal set of impact-parameter values (the fractal dimension of this set is $d\cong 0.67$ for this case[6]).

The relevant point here is that the scattering is regular for large E [Fig. 1(a)], but is chaotic at smaller E [Fig. 1(b)]. This is a very general feature of such problems. It is the purpose of this paper to investigate the possible "routes" to chaotic scattering. That is, how does chaos come about as the energy is continuously lowered? In our discussion it will be useful to distinguish what we call *fully developed* chaotic scattering. We use this term to denote a situation in which all periodic orbits are unstable and there are no Kolmogorov-Arnol'd-Moser (KAM) surfaces (i.e., the dynamics is hyperbolic). The general question of how chaos comes about has been extensively investigated for attractors, and has resulted in a number of often observed "scenarios" including period doubling, intermittency, crises, etc. Here we obtain the first answers to the question of how fully developed chaotic scattering arises. For the general class of two-dimensional scattering problems we consider, we find that fully developed chaotic scattering can appear in two possible ways depending on the form of the potential. In one of these transitions, the bifurcation is abrupt[7] in that the scattering is regular for E greater than a critical value E_c, but there is fully developed chaotic scattering as soon as E decreases through E_c. Also the number of unstable periodic orbits is some finite small value (zero or one in our examples) for $E>E_c$, but then jumps to infinity for $E<E_c$. [Chaos implies the presence of an infinite number of unstable periodic orbits such that the number of orbits with period less than some value T increases exponentially with T as $\exp(ST)$, where S is the topological entropy.[8]] We call this an *abrupt bifurca-*

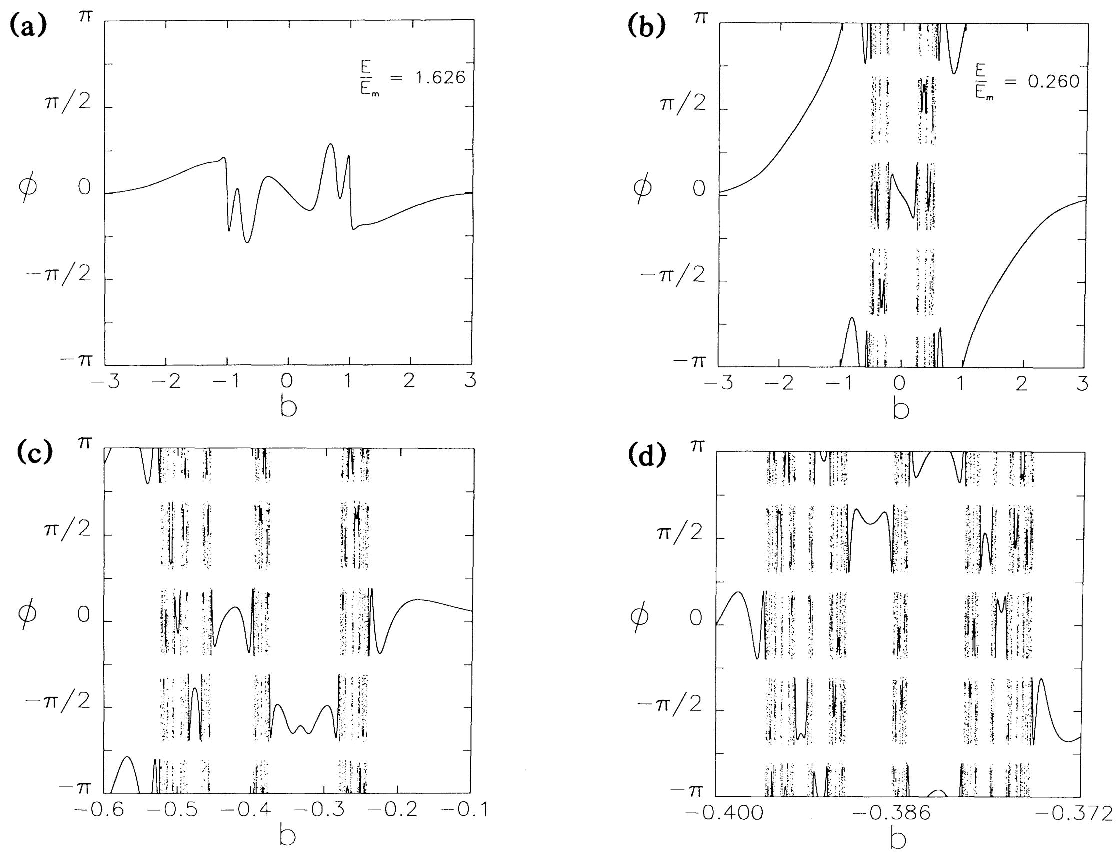

FIG. 1. Plot of deflection angle ϕ vs impact parameter b: (a) $E=1.626E_m$; (b) $E=0.260E_m$; (c) and (d) are blowups of (b).

tion to fully developed chaotic scattering, and we show that the fractal dimension in the scattering data has a characteristic nonanalytic scaling with energy for E near E_c. In contrast, for the second type of transition which we discuss, the creation of periodic orbits involves a sequence of saddle-node bifurcations and this implies the presence of KAM surfaces initially surrounding the stable nodes. In this case, when chaotic scattering first appears, it is not fully developed, but then becomes fully developed as E decreases further.

As background for our analysis, we now review the relevant facts concerning scattering from a single circularly symmetric monotonic potential hill. Figure 2 shows trajectories incident on such a hill for $E>E_m$ (where E_m is the maximum of the potential). As is evident from the figure, as the impact parameter decreases from large values, the angular deflection first increases, reaches a maximum angle $\phi_m<90°$, and then decreases. Further, it can be shown[6] that ϕ_m increases as E decreases toward E_m reaching a limiting value of $\phi_m=90°$ at $E=E_m+0^+$. (For $E<E_m$, $\phi_m=180°$, since the orbit with zero impact parameter is backscattered.)

The analysis and arguments are facilitated by assuming that the potential $V(\mathbf{r})$ consists of three hills whose separation is large compared to their widths. (These assumptions should not affect our main conclusions.[6]) We label these hills 1, 2, and 3 and denote the potential maxima at the hilltops by E_{m1}, E_{m2}, and E_{m3}, respectively, where, by convention, $E_{m3}\geq E_{m2}\geq E_{m1}$. We further assume that the potential for each isolated hill is monotonically decreasing from the hilltop (with a generic quadratic maximum) and that the potential is locally circularly symmetric about the hilltop. We distinguish two cases. Case 1 is shown in Fig. 3(a), and case 2 is shown in Fig.

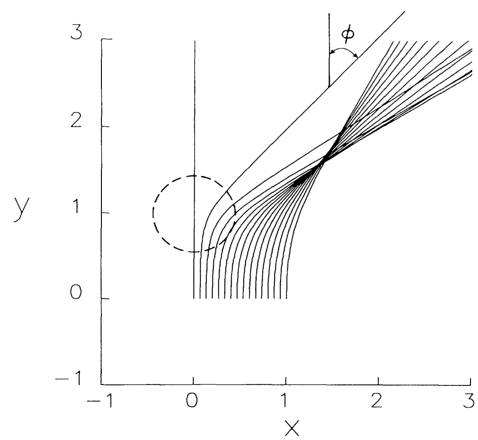

FIG. 2. Trajectories incident on a circularly symmetric potential hill for $E>E_m$.

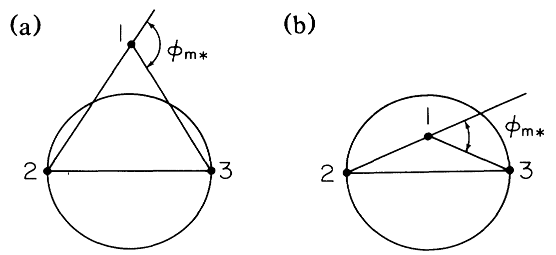

FIG. 3. Sketch illustrating two possible cases for three unequal hills. (a) Case 1 and (b) case 2.

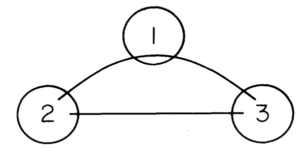

FIG. 4. Sketch of two possible orbits connecting hill 2 and hill 3 for case 2.

3(b). In case 1 [Fig. 3(a)] the hill of lowest maximum energy (hill 1 with maximum energy E_{m1}) is outside the circle whose diameter is the line joining the two hills of larger maximum energy, hills 2 and 3. In case 2 [Fig. 3(b)] hill 1 is inside this circle. (In both cases we presume that hill 1 is far from the circle in comparison with the width of a hill.)

Consider *case 1*. Say $E < E_{m2} \le E_{m3}$ and assume that our orbit is deflected from hill 2 (or hill 3) and travels toward hill 1. In order for this orbit to remain trapped it must be deflected back toward hill 2 or hill 3. Since hill 1 lies outside the circle, the minimum required deflection angle ϕ_{m*} is greater than 90°. Thus, recalling the result for a single hill, we see that for $E > E_{m1}$ there are no bounded orbits reflecting from hill 1. Consequently, the only periodic orbit that can exist is the one bouncing back and forth between hills 2 and 3. Thus there is no chaos for case 1 when $E > E_{m1}$. When E drops below E_{m1}, chaos is immediately created, since now the number of unstable periodic orbits increases exponentially with period: We can represent the periodic orbits as a sequence of symbols representing the order in which each hill is visited, and any sequence is possible. Thus, for case 1, we have an abrupt bifurcation to fully developed chaotic scattering. Note that the abrupt creation of chaos[7] as E decreases through the critical value $E_c = E_{m1}$ is accompanied by a change in the topology of the energy surface: For $E < E_{m1}$ a forbidden region $[V(\mathbf{r}) > E]$, where orbits cannot penetrate, is created about the maximum of hill 1.

Now consider *case 2* shown in Fig. 3(b), with energy in the range $E_{m1} < E < E_{m2} \le E_{m3}$. As E is decreased from E_{m2} to E_{m1} the maximum deflection hill 1 is capable of producing, $\phi_{m1}(E)$, increases monotonically from some value $\phi_{m1}(E_{m2}) < 90°$ to $\phi_{m1}(E_{m1}) = 90°$. Let ϕ_{m*} illustrated in Fig. 3(b) denote the deflection required by an orbit incident on hill 1 from hill 2 (respectively, hill 3) to be reflected toward hill 3 (respectively, hill 2). Now, however, since hill 1 is inside the circle, $\phi_{m*} < 90°$. We can now distinguish two subcases within case 2. Case 2(a): E_{m1} is small enough that $\phi_{m*} > \phi_{m1}(E_{m2})$. Case 2(b): $\phi_{m*} < \phi_{m1}(E_{m2})$.

In case 2(a), as E decreases, ϕ_{m1} will increase until at some value $E = E_{m*} > E_{m1}$, we have $\phi_{m*} = \phi_{m1}(E_{m*})$. For $E_{m1} < E \lesssim E_{m*}$ we can have orbits traveling back and forth between hills 2 and 3 in two ways: Either the path between hills 2 and 3 can pass through the region of hill 1 or it can bypass hill 1 going directly between hills 2 and 3. This is illustrated schematically in Fig. 4. Thus we expect that for E below (but not too close) to E_{m*} there will be unstable periodic orbits made up of all possible combinations of the two types of paths between hills 2 and 3 shown in Fig. 4. In this case we can represent the periodic orbits by all possible periodic sequences of two symbols (for the two types of elementary paths), and fully developed chaos is therefore present. The way in which this situation arises in this case is very different from what we have for case 1 and is *not* an abrupt bifurcation. In particular, as E decreases, producing chaos, there is no change in the topology of the energy surface. How can the infinite number of unstable periodic orbits necessary for chaos be created in this case? In the abrupt bifurcation it is the change in the energy surface topology that occurs when E passes through one of the E_{mi} which creates the infinity of periodic orbits. In the absence of such a change in topology, the only mechanisms available for the creation of unstable periodic orbits are the standard generic bifurcations of smooth Hamiltonian systems with two degrees of freedom: saddle-node bifurcations and period-doubling bifurcations. Note that due to the finite width of the hills the transition to fully developed chaotic scattering near $E \cong E_{m*}$ is not sharp. Thus as E decreases through a range near E_{m*}, there must be an intricate sequence of saddle-node bifurcations and period-doubling cascades. This type of sequence of events has been discussed in Ref. 9 for the dissipative case. Note that, in a saddle-node bifurcation for our Hamiltonian system, the nodes are stable elliptic orbits surrounded by KAM tori. As the energy is further lowered, we expect all stable elliptic orbits created in saddle-node bifurcations to be destroyed and replaced by unstable orbits via the mechanism whereby the nodes undergo period-doubling cascades.[9] When the process is completed all periodic orbits are unstable[9] (fully developed chaotic scattering), and follow all possible sequences of the two types of paths shown in Fig. 4. Numerical investigations confirming this phenomenology will be reported elsewhere.[10]

In case 2(b), as soon as E drops below E_{m2} we have a transition to fully developed chaotic scattering. This arises, since for any energy $E < E_{m2} \le E_{m3}$ we can have reflections from hills 2 and 3 at angles up to 180°, and,

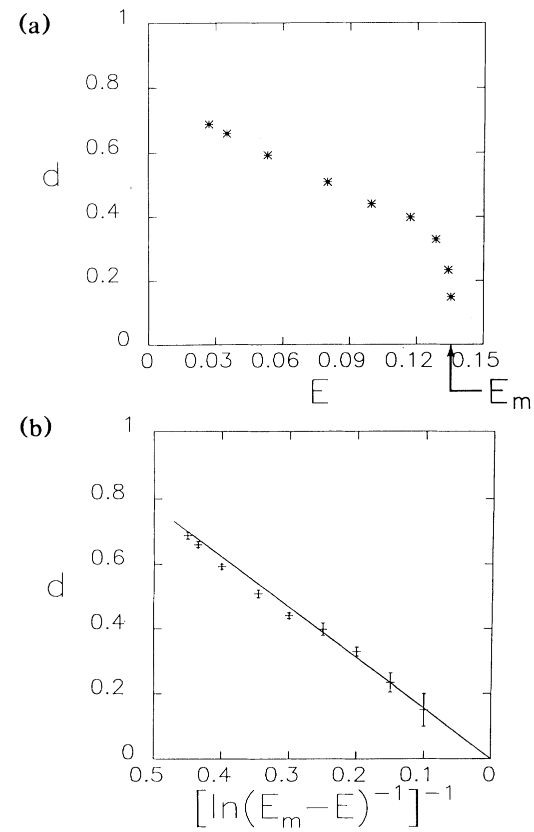

FIG. 5. Fractal dimension d vs energy. (a) Linear scale in energy. (b) Same data as in (a) plotted vs $[\ln(E_m-E)^{-1}]^{-1}$.

in addition, because $\phi_{m*} < \phi_{m1}(E)$ for $E < E_{m2}$, there are two possible paths between hills 2 and 3 (Fig. 4). Thus, for case 2(b), we have an abrupt bifurcation to chaotic scattering at $E = E_c = E_{m2}$.

We now discuss a quantitative characteristic feature of the abrupt bifurcation to fully developed chaotic scattering. In particular, we ask how does the fractal dimension d of the set of singular impact-parameter values in ϕ vs b plots [such as Fig. 1(b)] scale with the energy of the incident particle? In Ref. 6 it is shown analytically that d has a nonanalytic behavior near the critical energy E_c which is of the following form:

$$d \sim [\ln(E_c-E)^{-1}]^{-1}, \tag{1}$$

for $E < E_c$. As an example, we consider the square-symmetric, four-hill potential $V(\mathbf{r}) = x^2y^2\exp[-(x^2+y^2)]$ which was used for Fig. 1. A proof that this potential undergoes an abrupt bifurcation to fully developed chaotic scattering at $E_c = E_m$ is given in Ref. 6. Figure 5 shows plots of the fractal dimension d versus energy E. In Fig. 5(a) the energy scale is linear, and we note a sharp drop of d to zero at $E = E_m$ as would be predicted by the dependence in Eq. (1). Figure 5(b) shows the same data for d plotted versus $[\ln(E_m-E)^{-1}]^{-1}$. The result is an approximately linear behavior of d consistent with Eq. (1).

In summary, we have shown that, for a broad class of problems, fully developed chaotic scattering can come about in two ways. One of these transitions, which we call an abrupt bifurcation, leads to a characteristic scaling of fractal dimension with particle energy [Eq. (1)].

This research was supported by the Office of Naval Research (Physics) and by the Department of Energy (Basic Energy Sciences).

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[a]Permanent address: Department of Chemistry, University of Pennsylvania, Philadelphia, PA 19104.

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